TREATISE

ON

DIFFERENTIAL EQUATIONS.

SUPPLEMENTARY VOLUME.

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TREATISE

ON

DIFFERENTIAL EQUATIONS.

SUPPLEMENTARY VOLUME.

BY THE LATE

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PREFACE.

THE present volume contains all that Professor Boole wrote for the purpose of enlarging his Treatise on Differential Equations. Had he lived to publish the second edition he would doubtless have incorporated his more recent investigations with the original work, and it is therefore necessary to explain why another plan has been adopted.

In some cases Professor Boole had indicated that certain portions of the original work were to be omitted and their places supplied from the manuscripts; but on examination it appeared that in subsequent passages of the work there were references and allusions to the portions thus marked to be omitted which would not apply to the substituted matter. Thus in attempting to carry out the directions it would have been necessary to accept the responsibility of making many alterations, and consequently to incur the risk of failing in the attempt to improve the original form.

Moreover the Treatise had been for some time out of print, and the long delay which must have been caused by the labour of reconstruction would have produced serious inconvenience to students at Cambridge and elsewhere. Professor Boole himself was always especially anxious to consult the advantage of students, and those who had the charge of his manuscripts were naturally inclined to adopt a course of which they believed he would himself have approved.

The design of reconstructing the Treatise was therefore abandoned; and it was resolved that the original volume should be reprinted, and that the manuscripts should be collected and published separately. This plan has the obvious recommendation of enabling those who are already familiar with the original work to turn their attention readily to the new investigations. It will be seen that many of the Chapters of the present volume may be regarded as independent essays or memoirs which lose nothing by being separated from the other volume; and indeed no indications had been left by Professor Boole of the place which such Chapters were to occupy in the enlarged edition.

I have printed all the unpublished matter relating to Differential Equations which I found among Professor Boole's papers. In a few cases it will be seen that an investigation is incomplete; such investigations have however been included in the volume, because I was unwilling that anything should be lost which so great a mathematician had written on a subject he had long and carefully studied.

I trust that no serious error will be found in the volume, and that any faults which may be detected will be excused on account of the nature and difficulty of the task that had to be performed. Many of the manuscripts had not been finally revised; some of them were very obscure and had to be carefully and laboriously copied for the press. In general the equations were not numbered, and thus only blanks occurred in place of references; this circumstance often caused great trouble and perplexity: I hope however that a satisfactory result has been finally attained.

I may state for the benefit of those who are conversant with the first edition of the original work that the theorem which in the present volume is cited as contained in Chap. II. Art. 1 will be found in Chap. IV. Art. 2 of the first edition: the change was made by the direction of Professor Boole's interleaved copy. It was judged convenient to number the Chapters in the present volume in continuation of those in the original work.

All additions of my own are enclosed within square brackets. The sheets have been read by the Rev. J. Sephton, Fellow of St John's College, as well as by myself, and the volume is much indebted to his care and accuracy. Obvious mistakes in the manuscripts were of course corrected; thus, for example, the table at the end of the volume was calculated by Mr Sephton, because the table in the manuscript was rendered erroneous by the use of a wrong sign in a formula.

I. TODHUNTER.

ST JOHN'S COLLEGE,
November, 1865.

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On the Analysis of Discontinuous Functions. Vol. 21, 1848, pages 124...139.

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On the Transformation of Definite Integrals. Same Vol., pages 216...224.

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A Treatise on the Calculus of Finite Differences. Cambridge, 1860.

[This list contains all Professor Boole's writings which have fallen under the notice of the editor; it is possible that there may be a few omissions.]

CONTENTS.

CHAPTER XIX.

ADDITIONS TO CHAPTER II.

1. [In Chapter II. Art. 9, two methods are given for solving the differential equation

$$(ax + by + c)\, dx + (a'x + b'y + c')\, dy = 0.]$$

But there exists another transformation by which the equation may be reduced to, (because it may be constructed from), an equation in which the variables are separated.

Assume as this equation

$$(Ay' + C)\, dx' + (A'x' + C')\, dy' = 0 \ldots\ldots (1)$$

and let $$x' = x + m_1 y, \quad y' = x + m_2 y.$$

It will be seen that in these equations united we have as many constants as in the original equation. Now on substituting in the assumed equation the values of x' and y', and comparing with the equation given, we deduce a system of relations equivalent to the following, viz.:

The quantities m_1, m_2 are roots of the quadratic

$$am^2 - (b + a')\, m + b' = 0.$$

The quantities A, A', C, C' are determined by the system of equations

$$A + A' = a, \quad C + C' = c,$$

$$Am_1 + A'm_2 = a', \quad Cm_1 + C'm_2 = c',$$

from which we find

$$A=\frac{am_2-a'}{m_2-m_1},\qquad C=\frac{cm_2-c'}{m_2-m_1},$$

$$A'=\frac{am_1-a'}{m_1-m_2},\qquad C'=\frac{cm_1-c'}{m_1-m_2}.$$

Now (1) gives on dividing by $(A'x'+C')(Ay'+C)$ and integrating

$$\frac{1}{A'}\log(A'x'+C')+\frac{1}{A}\log(Ay'+C)=\text{const.},$$

or $$(A'x'+C')^{\frac{1}{A'}}(Ay'+C)^{\frac{1}{A}}=\text{const.},$$

which on substitution and reduction gives

$$\frac{\{(am_1-a')(x+m_1y)+cm_1-c'\}^{\frac{1}{am_1-a'}}}{\{(am_2-a')(x+m_2y)+cm_2-c'\}^{\frac{1}{am_2-a'}}}=\text{const.}\ldots\ldots(2)$$

2. Under certain circumstances the general solutions of differential equations of the first order fail. This happens in the above example if $m_2=m_1$, the solution then reducing to

$$1=\text{const.}$$

The theory of the deduction of the true limiting form of the solution in such cases requires a distinct statement.

Let the supposed general solution be represented by

$$u=C,$$

C being the arbitrary constant and u a function of x, y, and constants which are not arbitrary. Suppose too that when one of these constants k assumes a particular value κ, the function u reduces to a constant v. Then we have

$$\frac{u-v}{k-\kappa}=\frac{C-v}{k-\kappa}.$$

Now the second member being a function of an arbitrary constant is equivalent to an arbitrary constant and may be replaced by C. The first member is a vanishing fraction, the limiting value of which is $\left(\frac{du}{dk}\right)$, the brackets being used to denote that after the differentiation k is to be made equal to κ. Hence the solution becomes

$$\left(\frac{du}{dk}\right) = C.$$

In applying this theory to the reduction of the general solution (2) in the case in which $m_1 = m_2$, it must be observed that the numerator of the first member is the same function of m_1, x, y, as the denominator is of m_2, x, y; or attending solely to their functional character with respect to m_1, m_2, we may affirm that the numerator is the same function of m_1 as the denominator is of m_2. Representing these functions by $\phi(m_1)$, $\phi(m_2)$ respectively, we have

$$u = \frac{\phi(m_1)}{\phi(m_2)}.$$

But m_1, m_2 being roots of a quadratic equation may be represented in the form

$$m_1 = m + k, \quad m_2 = m - k,$$

the roots becoming equal when $k = 0$. Hence

$$u = \frac{\phi(m+k)}{\phi(m-k)}.$$

Therefore since

$$\frac{d\phi(m+k)}{dk} = \frac{d\phi(m+k)}{dm}, \qquad \frac{d\phi(m-k)}{dk} = -\frac{d\phi(m-k)}{dm},$$

we have

$$\frac{du}{dk} = \frac{\phi(m-k)\dfrac{d\phi(m+k)}{dk} - \phi(m+k)\dfrac{d\phi(m-k)}{dk}}{\{\phi(m-k)\}^2};$$

therefore
$$\left(\frac{du}{dk}\right) = \frac{2\phi(m)\dfrac{d\phi(m)}{dm}}{\{\phi(m)\}^2} = \frac{2\dfrac{d\phi(m)}{dm}}{\phi(m)}$$
$$= 2\frac{d}{dm}\log\phi(m).$$

Thus the solution becomes on putting C for $\frac{C}{2}$,

$$\frac{d}{dm}\log\phi(m) = C,$$

or
$$\frac{d}{dm}\frac{1}{am-a'}\log\{(am-a')(x+my)+cm-c'\} = C.$$

3. [The next Article seems to have been intended to appear in the enlarged form of Chap. II.; but I cannot discover what precise position it would have occupied. I conjecture that "the above demonstration" refers to Chap. II. Arts. 2, 3; and I have accordingly supplied a reference to equation (3) of Chap. II.

I had myself drawn Professor Boole's attention to Chap. II. Arts. 2, 3. The geometrical process of Chap. II. Art. 3, appears to have been first given by D'Alembert in his *Opuscules*, Vol. IV. p. 255. D'Alembert calls it a *demonstration;* it seems to me only an *illustration*, at least in the brief form of the text: and that such was Cauchy's opinion may perhaps be inferred from the elaborate investigation given by Moigno, to which Professor Boole refers in Art. 5 of the present Chapter.

I had also drawn Professor Boole's attention to the statement at the end of Chap. II. Art. 12, that only *one* arbitrary constant was involved. Accordingly Article 5 of the present Chapter developes this statement, and Article 4 seems intended to bear on the same subject.]

4. In the above demonstration the relation between y and x is regarded as one of pure magnitude, and the interpretation of the differential equation becomes a limiting case of that of the equation of *finite* differences (Eq. (3), Chap. II.). But if we represent x and y by the rectangular co-ordinates

of a moving point on a plane the differential equation may be interpreted directly. For supposing it reduced to the form

$$\frac{dy}{dx} = f(x, y),$$

we see that the direction of motion is constantly assigned as a function of the co-ordinates of position. The entire motion is therefore determinate as soon as the initial point is fixed. The result of the motion is a line or curve wholly continuous or subject to irregularities according to the nature of the function $f(x, y)$. That the arbitrariness of origin is geometrically equivalent to the appearance of a single arbitrary constant in the relation connecting x and y may be shewn thus.

Let $$y = \phi(x_0, y_0, x)$$

be the relation between x and y indicated by the supposed motion, x_0, y_0 being the initial point of departure. Then this point being on the line of motion, x_0, y_0 are particular values of x and y, so that we have from the above equation

$$y_0 = \phi(x_0, y_0, x_0),$$

which establishes a relation between x_0 and y_0, and shews that there exists virtually but one arbitrary constant.

5. It is proved in Art. 3, Chap. II., that the constants x_0, y_0, initial values of the variables x, y in the solution of the differential equation of the first order, are necessarily equivalent to one *arbitrary* constant. I shall shew from the form of the above solution that this *a priori* condition is actually satisfied.

Developing the expression for y [see Eq. (30) of Chap. II.] in ascending powers of x, we have

$$y = A_0 + A_1 x + \frac{A_2 x^2}{1.2} + \frac{A_3 x^3}{1.2.3} + \&c., \quad \text{........} (32)$$

in which $$A_r = \Sigma \frac{f_n(x_0, y_0)(-x_0)^{n-r}}{1.2 \dots (n-r)},$$

the summation extending from $n = r$ to $n = \infty$. Forming hence the differential coefficients of A_r with respect to x_0 and y_0, and reducing by (28), we shall find

$$\frac{dA_r}{dx_0} + f_1(x_0, y_0)\frac{dA_r}{dy_0} = 0,$$

whence in particular

$$\frac{dA_0}{dx_0} + f_1(x_0, y_0)\frac{dA_0}{dy_0} = 0.$$

Eliminate between these equations $f_1(x_0, y_0)$, and we have

$$\frac{dA_r}{dx_0}\frac{dA_0}{dy_0} - \frac{dA_r}{dy_0}\frac{dA_0}{dx_0} = 0.$$

Therefore, by Prop. I., A_r is a function of A_0, so that the solution reduced to the form (32) contains but the single arbitrary constant A_0.

It remains to notice that the solution must be applied only under the conditions of convergency, i.e. under the condition that the ratio of the n^{th} to the $(n-1)^{\text{th}}$ term tends to a limit less than unity as n tends to infinity. For a discussion of the failing cases of this test see 'Finite Differences,' Chap. V. Generally it is desirable, in order to secure rapid convergency, to divide the interval $x - x_0$ into separate equal portions, to each of which the general theorem of solution may be applied. If $x - x_0$ be very small the theorem may be approximately represented by

$$y - y_0 = f(x_0, y_0)(x - x_0).$$

On these principles Cauchy has founded remarkable methods of solution, which deserve attention from the commentary on the limits of error on their application by which they are accompanied (Moigno, Vol. II. pp. 385—434).

CHAPTER XX.

ADDITIONS TO CHAPTER VII.

1. [THIS Article relates to Art. 2 of Chap. VII.]

The sense in which (9) may be said to constitute the general solution of the differential equation is this. We obtain from it

$$y = C\epsilon^{ax}, \qquad y = C\epsilon^{-ax};$$

giving any particular value to C this will geometrically represent a curve consisting of two branches, and giving to C every possible value we obtain an infinite system of such curves, each consisting of two branches. The aggregate of branches thus obtained is evidently the same as the aggregate of curves given by the two primitives (5) and (6), unrestricted by any connexion between c_1 and c_2. In this sense then the solution (9) is general, that it includes all the particular relations between y and x which are deducible from the original primitives (5) and (6). And it is only in this sense not general that it groups these relations together in a particular manner.

To the expression of the complete primitive a certain variety of form may be given without affecting its generality in the sense above affirmed. Thus, if to the solutions of the component differential equations we give the forms

$$y\epsilon^{-ax} - c_1 = 0, \qquad \log y + ax - c_2 = 0,$$

we should have, by the same procedure, as the expression of the complete primitive,

$$(y\epsilon^{-ax} - c)(\log y + ax - c) = 0,$$

an equation which may equally with (9) be regarded as the complete primitive of the differential equation given, and which in geometry represents the same totality of branches of curves as (9), with this difference only, that they are differently paired together.

2. [This Article relates to Art. 3 of Chap. VII.]

The question will here naturally arise, Since if $V = c$ be a solution of one of the component differential equations, $f(V) = c$, in which $f(V)$ denotes any function of V, is also a solution, by Chap. IV. Art. 3, why not give to the complete primitive the form

$$\{f_1(V_1) - c\}\{f_2(V_2) - c\}\ldots\ldots\ldots\{f_n(V_n) - c\} = 0,$$

or the stricter form

$$f_1(V_1)f_2(V_2)\ldots\ldots\ldots f_n(V_n) = 0 \ldots\ldots\ldots\ldots(F),$$

in which $f_1(V_1)$, $f_2(V_2)$, ... $f_n(V_n)$ denote arbitrary functions of V_1, V_2,..., V_n respectively—stricter because the presence of arbitrary constants and functions in the previous form is a superfluous generality? It is replied that though the form just given is analytically more general than (15), it is not more general than (15) with such freedom as is permitted in the interpretation of the arbitrary constants. In a physical or geometrical application we should not only be permitted to assign a particular value to the arbitrary constant in (15), so deducing what in reference to its source would then be termed a particular primitive, but to combine the results of different determinations of c together, so as to obtain every form of solution which is implied either in the functional equation (F), or in its component primitives

$$V_1 = c_1, \quad V_2 = c_2 \ldots, \quad V_n = c_n.$$

The same considerations justify us in speaking of (15) as the complete primitive, and not as a complete primitive.

CHAPTER XXI.

ADDITIONS TO CHAPTER VIII.

1. [THE Singular Solutions of Differential Equations of the First Order received great attention from Professor Boole, and the Chapter devoted to that subject is one of the most valuable and important in his work. He continued his researches after the publication of his first edition, and intended to reconstruct the Chapter with great improvements in the second edition. After carefully examining the manuscripts I came to the conclusion that it would be very difficult to rewrite this portion of the work so as to connect the old matter with the new; and thus it seemed best to reprint the original Chapter VIII. with corrections of obvious misprints, and to print the matter intended for the revised form in the present volume. The plan gives rise to some repetition; but this seems unimportant, compared with the advantage of preserving in the author's own language all that he left on an interesting and important point which he had carefully studied.

2. It may be of service to the student to reproduce the substance of some remarks on his Chapter VIII. which were sent to Professor Boole soon after the publication of his first edition; for there is evidence in his manuscripts that he paid great attention to such remarks while engaged in the revision of his work, and thus the reason and the meaning of some of his additions and changes may be made more obvious. These remarks will occupy the next Article.

3. The two pages beginning with "And these conditions are sufficient," and ending with "do not lead to *conflicting*

results," forming part of Arts. 3 and 4 of Chapter VIII., seem obscure and difficult. The following may perhaps be substituted with advantage.

The only ways in which

$$\frac{dy}{dx} = \frac{df(x, c)}{dx} \text{ and } \frac{dy}{dx} = \frac{df(x, c)}{dx} + \frac{df(x, c)}{dc}\frac{dc}{dx}$$

can be equivalent when c is variable, are

(1) when $\frac{df(x, c)}{dc} = 0$,

(2) when $\frac{df(x, c)}{dx} = \infty$;

in the latter case $\frac{dy}{dx} = \infty$, and therefore $\frac{dx}{dy} = 0$, and this implies that the singular solution is of the form $x = \text{constant}$. Thus there can be no singular solutions except such as are found from $\frac{df(x, c)}{dc} = 0$, and such as are found from $x = \text{constant}$.

Similarly, if the complete primitive be expressed in the form $x = F(y, c)$, there can be no singular solutions except such as are found from $\frac{dF(y, c)}{dc} = 0$, and such as are found from $y = \text{constant}$.

In Art. 8 of Chapter VIII. we read, "We may pass over the case in which the above equation is satisfied independently of c, because the relation obtained would involve x only, while it is a condition accompanying the use of $\frac{dp}{dy} = \infty$ that it leads to solutions involving y at least." It is objected, Why may we pass over this case? Such a case might occur and furnish a solution, and then we should want to know the character of that solution. Take for example $p = x^n y$; here if n is negative, $\frac{dp}{dy}$ is infinite when $x = 0$, and this is a singular solution. For the general solution is $y = ce^{\frac{x^{n+1}}{n+1}}$, and so $x=0$ is not a case of it. The words—*while it is a condition...at least*—seem very difficult, for by supposition we are now investigating what is furnished by $\frac{dp}{dy}=\infty$.

Professor Boole met the objection in substance thus:

"It will be found that the rules in the book are correct in this case. What is implied in the Chapter, though not stated with sufficient clearness, is that if $\frac{dp}{dy}=\infty$ leads to a solution which does not involve y in its expression, nothing is to be inferred whether it is singular or not. Then the proper test is $\frac{d}{dx}\left(\frac{1}{p}\right)=\infty$.

In this example we have

$$\frac{dp}{dy}=\infty \text{ gives } x^n=\infty \text{ ; no inference;}$$

$$\frac{d}{dx}\left(\frac{1}{p}\right)=\infty \text{ gives } x^{-(n+1)}y^{-1}=\infty.$$

Hence $x=0$, provided n is between 0 and -1, or $y=0$.

Consider these separately:

First. Let n be between 0 and -1, and $x=0$. This is by the test a singular solution. Substituting it in the complete primitive we get $y=c$, which confirms this.

Second. Let $y=0$. This satisfies the differential equation; but from the fact that it comes from $\frac{d}{dx}\left(\frac{1}{p}\right)=\infty$ we have no inference; from the fact that it does not come from $\frac{dp}{dy}=\infty$ we have the inference that it is a particular integral: it corresponds to $c=0$.

There remains the case of $x=0$ when n is between -1 and $-\infty$. As this does not satisfy $\frac{d}{dx}\left(\frac{1}{p}\right)=\infty$, we infer that it is a particular integral. To prove this we have

$$c=y\epsilon^{-\frac{x^{n+1}}{n+1}}.$$

When $x=0$ this gives, since $1+n$ is negative,

$$c=\infty \text{ or } c=-\infty,$$

according as y is positive or negative. This is like Ex. 2 of Chap. VIII. Art. 8."

The remark made by Professor Boole in the above reply, that *if* $\frac{dp}{dy}=\infty$ *leads to a solution which does not involve* y *nothing is to be inferred*...is important. It corrects the statement put too strongly in Chap. VIII. Art. 7, "All we can affirm is that if $\frac{dp}{dy}=\infty$ gives a solution at all it will be a singular solution."

From Art. 8 onwards it seems assumed that a solution for which $\frac{dy}{dc}=0$ is always to count as a singular solution, even if it should coincide with a particular integral. This does not seem to have been quite the view of the former part of Chapter VIII.: see Arts. 5 and 6 of the Chapter.

In Ex. 3 of Art. 9 we read, "the second is obviously a singular solution." This means that since we have a solution which makes $\frac{dp}{dy}$ infinite, we conclude that it is a singular solution.

So in Ex. 5 of Art. 11 we read, "is evidently a singular solution," when it seems better to say, "and is therefore a singular solution."

4. The additional matter relating to Chapter VIII. begins with another example which was to be placed at the close of Art. 3 of that Chapter.]

Ex. The differential equation

$$(\sqrt{x^2+y^2-m^2}-y)\frac{dy}{dx}-x=0$$

has for its complete primitive

$$\sqrt{x^2+y^2-m^2}-y-c=0.$$

Here $\dfrac{d\phi}{dy}=\dfrac{y}{\sqrt{x^2+y^2-m^2}}-1, \quad \dfrac{d\phi}{dx}=\dfrac{x}{\sqrt{x^2+y^2-m^2}},$

$$\frac{d\phi}{dc}=-1.$$

Hence $\dfrac{dy}{dc}=\dfrac{\sqrt{x^2+y^2-m^2}}{y-\sqrt{x^2+y^2-m^2}}, \quad \dfrac{dx}{dc}=\dfrac{\sqrt{x^2+y^2-m^2}}{x}.$

Both $\dfrac{dy}{dc}$ and $\dfrac{dx}{dc}$ vanish then if

$$x^2+y^2-m^2=0.$$

This therefore is the singular solution and it satisfies both the tests, as both x and y are contained in its expression.

Of the partial tests

$$\frac{d\phi}{dc}=0, \quad \frac{d\phi}{dx}=\infty, \quad \frac{d\phi}{dy}=\infty,$$

the first is not satisfied, the last two are satisfied.

The determination of c as a function of x by the solution of the equation $\dfrac{df(x, c)}{dc}=0$ is equivalent to determining what particular primitive has contact with the envelope at that point of the latter which corresponds to a given value of x.

One important remark yet remains. The elimination of c between a primitive $y=f(x, c)$ and the derived equation $\dfrac{dy}{dc}=0$, does not *necessarily* lead to a singular solution in the sense above explained. For it is possible that the derived equation

$$\frac{df(x, c)}{dc} = 0$$

may neither on the one hand enable us to determine c as a function of x, so leading to a singular solution; nor, on the other hand, as an absolute constant, so leading to a particular primitive. Thus the particular primitive

$$y = e^{cx}$$

being given, the condition $\frac{dy}{dc}$ gives

$$e^{cx} = 0,$$

whence c is $+\infty$ if x be negative, and $-\infty$ if x be positive. It is a dependent constant. The resulting solution $y = 0$ does not then represent an envelope of the curves of particular primitives, nor strictly one of those curves. It represents a curve formed of branches from two of them. It is most fitly characterized as a particular primitive marked by a singularity in the mode of its derivation from the complete primitive.

All the foregoing observations and conclusions may be extended to the case of solutions derived from the condition $\frac{dx}{dc} = 0$.

5. We have seen that the equation $\frac{dy}{dc} = 0$ may be satisfied by an absolutely constant value of c, so leading to a particular primitive and not a singular solution. In this case $\psi(x + h, c)$ as well as $\psi(x, c)$ would vanish, and the numerator of (9), instead of being the difference of a finite and an infinite quantity, would be the difference of two infinite and equal quantities. [See Chap. VIII. Art. 8.] It would not therefore be infinite. Hence we conclude that $\frac{dp}{dy}$ would not become infinite for a particular primitive in the strict sense of that

term, i. e. for a solution derived from the complete primitive by giving to c an absolutely constant value.

This is one point of contrast between the conditions

$$\frac{dy}{dc}=0, \quad \frac{dp}{dy}=\infty.$$

There is another not less important. As the numerator of (9) may become infinite not only when $\psi(x, c)=0$, but also when $\psi(x, c)=$ infinite, we see that a relation between y and x which makes $\frac{dp}{dy}$ infinite will not necessarily satisfy the differential equation. On the other hand, it is not a particular primitive in the strict sense of that term.

Exactly in the same way the condition $\frac{dx}{dc}=0$, as relating to the complete primitive, leads to the condition

$$\frac{d}{dx}\left(\frac{1}{p}\right)=\infty,$$

as relating to the differential equation, with the same points of difference in the respective applications.

Ex. Let $\frac{dy}{dx}=my^{\frac{m-1}{m}}$, and suppose m a positive constant greater than 1.

Here $$\frac{dp}{dy}=(m-1)\,y^{-\frac{1}{m}},$$

which becomes infinite when $y=0$. As this involves y and satisfies the differential equation it is a singular solution.

To confirm this conclusion we may refer to the complete primitive

$$y=(x-c)^m,$$

which does not give $y=0$ for any particular value of c.

Now let m be a *positive* constant less than 1. We have still $\frac{dp}{dy}=\infty$ when $y=0$; but this value of y no longer satisfies the differential equation. It is not a solution at all, nor would it result from the application of the condition $\frac{dy}{dc}=0$ to the complete primitive. The distinction of character of the two tests is here made manifest.

6. We may express the most important results of the foregoing investigations in the following theorem.

THEOREM. Every solution of a differential equation of the first order which is derived from the complete primitive by giving to c a variable value will, if it involve y in its expression, satisfy the condition

$$\frac{dp}{dy}=\infty;$$

and if it involve x, the relation

$$\frac{d}{dx}\left(\frac{1}{p}\right)=\infty.$$

But relations satisfying these conditions will not necessarily be solutions of the differential equation.

In applying this theorem the following points must be carefully attended to.

1st. No conclusion can be drawn from the satisfying of the condition $\frac{dp}{dy}=\infty$ when the relation in question does not contain y in its expression, nor from the satisfying of

$$\frac{d}{dx}\left(\frac{1}{p}\right)=\infty$$

when the relation in question does not involve x in its expression. For these conditions being respectively derived from $\frac{dy}{dc}=0$ and $\frac{dx}{dc}=0$ are subject to the same limitations in their application.

2ndly. It may be that $\frac{dp}{dy}$ or $\frac{d}{dx}\left(\frac{1}{p}\right)$ assumes for a particular relation between x and y the indefinite form $\frac{0}{0}$. In this case we must seek by the development of its terms or by other known modes its true limiting value or values. Finite values will indicate particular primitives, infinite values singular solutions, and when such values emerge together out of the same relation between the variables, the solution will be a particular primitive possessing the geometrical properties of a singular solution. Its locus will be a particular curve enveloping other curves of the same family.

See Examples 2 and 3 of Chap. VIII. Art. 11.

We have seen that the conditions

$$\frac{dp}{dy}=\infty, \qquad \frac{d}{dx}\left(\frac{1}{p}\right)=\infty$$

indicate in general the existence of a relation between c and x or c and y. And when that relation is such as to enable us to determine c as a continuous function of one of the variables, the corresponding solution of the differential equation is singular, and is geometrically represented by an envelope of the curves of primitives. But it may be, as we have seen in a particular example, that the relation does not determine c as a function of x or y; but according to the language already used, c is a dependent constant, or in some other way different from the constant of an ordinary particular primitive. Let us examine in particular instances the kind of singularity which may hence arise.

Ex. 1. Given $p=\dfrac{y\log y}{x}$.

Here $$\frac{dp}{dy}=\frac{1}{x}(1+\log y).$$

This becomes infinite if $x=0$; but this not involving y must be rejected. Again, it becomes infinite if $y=0$, and this proves to be a solution of the differential equation, the limiting value of the indeterminate function in the second member being 0 (Todhunter's *Differential Calculus*, Chap. X.). Now the complete primitive is $y=e^{cx}$, discussed in Art. 4. The constant c is there shewn to be dependent, the solution $y=0$ emerging from the complete primitive by making $c=-\infty$ if x be positive, and $c=\infty$ if x be negative.

Ex. 2. Given $\left(\frac{dy}{dx}\right)^2 - xy\frac{dy}{dx} + y^2 \log y = 0.$

Here $$p=\frac{xy \pm y(x^2-4\log y)^{\frac{1}{2}}}{2};$$

therefore $$\frac{dp}{dy}=\frac{x \pm (x^2-4\log y)^{\frac{1}{2}}}{2} \mp \frac{1}{(x^2-4\log y)^{\frac{1}{2}}},$$

and this is made infinite by $y=0$ and by $x^2-4\log y=0$, i.e. by

$$y=0, \quad y=\epsilon^{\frac{x^2}{4}}.$$

Both satisfy the differential equation.

Now the complete primitive is

$$y=\epsilon^{cx-c^2}.$$

We see at once therefore that the second of the above solutions is singular. The first however is deducible from the complete primitive by making $c=\infty$ or $c=-\infty$, irrespectively of the *sign* or value of x, *provided only that x be finite*; not so however if x be infinite. The value of c is not therefore in the most absolute sense independent of that of x. If from the complete primitive we seek the singular solution by the condition $\frac{dy}{dc}=0$, we get the two equations

$$\epsilon^{cx-c^2}=0, \quad x-2c=0.$$

The second of these determines c as a function of x, and leads to the second of the solutions obtained above. The first, though it does not determine c as a function of x, still expresses a *relation* between c and x, which is the ground of the fulfilment of the condition

$$\frac{dp}{dy}=\infty.$$

We may further notice a peculiarity arising from this relation. Supposing x finite and the solution $y=0$ a particular integral, it presents the singularity that it is the only case in which two particular integrals agree. We might in any complete primitive, by changing c into c^2, get two values of c for the same particular integral, but then it would be for *every* particular integral.

One negative character seems indeed to mark all the cases in which a solution involving y in its expression satisfies the condition $\frac{dp}{dy}=\infty$. It is that such solutions do not emerge from the complete primitive by the attributing of a single and absolutely constant value to c. The relation which makes $\frac{dp}{dy}$ infinite satisfies the differential equation only because it satisfies the condition $\frac{dy}{dc}=0$, and this implies a connexion between c and x, which is the ground of a real though it may be unimportant singularity in the solution itself.

At this point, then, the question arises, whether the term singular solution shall be confined to that class of solutions, the loci of which represent the envelopes of curves of primitives, or shall be extended to all solutions which, satisfying the condition $\frac{dp}{dy}=\infty$, indicate the existence of a relation between c and x, and possess an actual singularity arising from this source. While the all but universal consent of mathematicians is in favour of the former course, it is to be remembered that the question is solely one of *definition*. Not such is the question how singular solutions of the envelope species, or as would more generally be said true singular solutions, are to be distinguished from all other solutions. This we now propose to consider. The question is not an isolated one. It stands in close relation to a series of properties of singular solutions which admit of an orderly development.

Discrimination of singular solutions of the envelope species.

7. A negative test, which in the great majority of cases suffices for the present object, is suggested by the following consideration.

The differential equation determining $\frac{dy}{dx}$ as a function of x and y determines also $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, *ad inf.*, and the knowledge of these enables us to construct in a developed form the complete primitive. See Chap. II. Art. 12.

The values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. *ad inf.*, as derived from the differential equation, are the same as those derived from the complete primitive.

But a solution deduced from the condition $\frac{dp}{dy} = \infty$ is only constructed so as to yield the same value of $\frac{dy}{dx}$ as the given differential equation does. If it be of the envelope species, the curve it represents has in general no continuous contact with the curve of any particular primitive. It will not therefore *generally* yield the same values for $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. as the differential equation does. It will not therefore generally satisfy the differential equations of an order higher than the first, which would be derived from the given equation by differentiation. Hence we have the following Proposition.

PROPOSITION. *If a relation which makes* $\frac{dp}{dy}$ *infinite satisfy the given differential equation of the first order, but do not satisfy all the higher differential equations obtained from it, such solution will be singular and of the envelope species.*

Ex. 1. By comparison with its complete primitive we saw in Art. 5 that $\frac{dy}{dx} = my^{\frac{m-1}{m}}$ has for a singular solution $y = 0$ when m is a constant greater than 1.

We will first suppose m a fractional quantity greater than 1, and endeavour to deduce the character of the solution without making use of the complete primitive.

From the solution we have

$$\frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0, \text{ \&c. } ad\ inf.$$

But from the differential equation

$$\frac{d^2y}{dx^2} = (m-1)\, y^{-\frac{1}{m}} \frac{dy}{dx} = m\,(m-1)\, y^{\frac{m-2}{m}},$$

and generally

$$\frac{d^ry}{dx^r} = m\,(m-1)\ldots(m-r+1) y^{\frac{m-r}{m}}.$$

Hence, when r is less than m, the substitution of $y = 0$ gives

$$\frac{d^ry}{dx^r} = 0$$

as before. But if r is greater than m, it gives

$$\frac{d^ry}{dx^r} = \infty.$$

We conclude that the solution is of the envelope species.

Secondly, suppose m a positive integer greater than 1.

In this case we find, when r is less than m, the same series of values as before; but for $r = m$ we have

$$\frac{d^ry}{dx^r} = m\,(m-1)\ldots 1,$$

and this also shews the solution to be of the envelope species.

Ex. 2. The differential equation

$$\frac{(y-xp)^2}{1+p^2}=\frac{(x^2+y^2)^2}{4}$$

is satisfied by

$$x^2+y^2=4.$$

Is this a singular solution or a particular integral?

From the solution we find

$$\frac{dy}{dx}=-\frac{x}{y},\quad \frac{d^2y}{dx^2}=-\frac{4}{y^3}.$$

From the differential equation we shall have

$$\frac{d^2y}{dx^2}=-\frac{(1+p^2)^2(x^2+y^2)}{2(y-xp)},$$

substituting in which the value of $\frac{dy}{dx}$, obtained from the proposed solution, we find

$$\frac{d^2y}{dx^2}=-\frac{(x^2+y^2)^2}{2y^3}=-\frac{8}{y^3}.$$

Now this differing from the value before obtained, we conclude that the solution is singular and of the envelope species.

And this result is verified by comparing the solution with the complete primitive

$$(x-c)^2+(y-\sqrt{1-c^2})^2=1.$$

As the test above exemplified is merely negative, it is insufficient. For it is conceivable that an enveloping curve should have an infinite order of contact with each of the curves which it envelopes, and this is also possible. Any test founded upon a comparison of the values of differential coefficients, any test therefore furnished by the *Differential* Calculus, would be insufficient for the discrimination of such cases.

Ex. 3. Given $\frac{dy}{dx}=y(\log y)^2$.

Here $\frac{dp}{dy} = \infty$ gives $y = 0$, and this satisfies the differential equation.

From this solution we find

$$\frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0, \text{ \&c. } ad\ inf.$$

From the differential equation we have

$$\frac{d^2y}{dx^2} = y\{(\log y)^4 + 2(\log y)^3\},$$

which consists of y multiplied by a rational and entire function of $\log y$. It is easy to see that all the higher differential coefficients of y hence derived will possess the same character. And all such vanish with y.

We can therefore neither affirm nor deny that the proposed solution is of the envelope species.

8. Before demonstrating a general Rule for the discrimination of solutions of this character, we shall notice certain of their properties which serve to indicate in what direction the Rule is to be sought. [See Chap. VIII. Art. 14.]

As the exact differential equation differs from the supposed given differential equation by having acquired a factor which the singular solution makes infinite, so the given differential equation may be said to differ from the corresponding exact one by containing a factor which the singular solution makes to vanish. If we knew that factor, we could by rejecting it reduce the given differential equation to a form in which it would no longer be satisfied by the singular solution. Now Poisson has shewn on a particular assumption, which does not however affect the principle of the demonstration, that this factor can be found when the singular solution is known. His demonstration is in substance as follows.

Let us represent the given singular solution of the differential equation by

$$u = 0,$$

u being a given function of x and y. Then introducing u and

x instead of y and x as variables, the differential equation after transformation will assume the form

$$\frac{du}{dx} = f(x, u).$$

Now this equation being satisfied by $u = 0$ and the first member vanishing, the second must also. Poisson now assumes, and the assumption must be carefully noted, this second member to be capable of being developed in ascending positive powers of u. Supposing it so developed, the differential equation becomes

$$\frac{du}{dx} = Au^{\alpha} + Bu^{\beta} + \&c.,$$

in which A, B,... are functions of x, and α, β,... ascending positive indices.

Hence if $u = 0$ be a singular solution we have, putting p for $\frac{du}{dx}$,

$$\frac{dp}{du} = A\alpha u^{\alpha-1} + B\beta u^{\beta-1} + \&c. = \infty.$$

But this demands that there should be at least one negative power of u in the development in the second member. Therefore $\alpha - 1$, the lowest index, must be negative. Therefore α being already positive must lie between 0 and 1.

We may give therefore to the transformed differential equation the form

$$\frac{du}{dx} = Qu^{\alpha},$$

α being a positive fraction, and Q not vanishing with u. Hence, dividing by u^{α},

$$u^{-\alpha}\frac{du}{dx} = Q,$$

or

$$\frac{1}{1-\alpha}\frac{d}{dx}u^{1-\alpha} = Q,$$

a differential equation which is *not* satisfied by $u=0$, since $u=0$ gives $u^{1-n}=0$, and the first member vanishes while the second member does not vanish. In its present form then the equation is not satisfied by $u=0$. We see also that the property of being satisfied by $u=0$ has been lost not in reality through a transformation, but through the rejection of an algebraic factor u^n from the transformed equation. It has been shewn in the treatment of Clairaut's equation, how in the ascent by differentiation to an equation of a higher order a somewhat analogous effect is produced, the singular solution emerging out of a factor of that higher equation.

If we inquire what is essential in Poisson's demonstration, we shall find it to consist in that the transformed equation is of the form

$$\frac{du}{dx}=QU,$$

in which while Q neither vanishes nor becomes infinite when $u=0$, the functions

$$U \text{ and } \int_0^u \frac{du}{U}$$

both vanish with u. The question whether U is of the form u^n as Poisson supposes, or is not, is wholly immaterial. This will fully appear from the demonstration of the following theorem, which is in effect Poisson's freed from arbitrary assumptions.

9. PROPOSITION. *If $u=0$ be a solution of a differential equation of the first order between y and x, and*

$$\frac{du}{dx}=f(x, u)$$

represent the form which that equation assumes when u and x are assumed as variables instead of y and x, then if $f(x, u)$ be resolved into two factors Q, U, of which Q neither vanishes nor becomes infinite when $u=0$, while the functions U and $\int_0^u \frac{du}{U}$ both vanish when $u=0$, then the differential equation can be reduced to a form in which it shall cease to be satisfied by $u=0$.

In the statement of this proposition x is supposed to be constant in the integration relative to u.

The differential equation after the transformation which introduces u and x as variables becomes

$$\frac{du}{dx} = QU.$$

Let
$$\int_0^u \frac{du}{U} = v,$$

so that v is in general a function of x and u, the form of which is known by integration when that of U is given. And again, transform the differential equation by making v and x the variables instead of u and x. We have

$$\left(\frac{dv}{dx}\right) = \frac{dv}{dx} + \frac{dv}{du}\frac{du}{dx},$$

in which $\frac{dv}{dx}$ is the differential coefficient of v with respect to x, on the above hypothesis as to the constitution of v as a function of x and u, while $\left(\frac{dv}{dx}\right)$ is the differential coefficient on the hypothesis that v is reduced to a function of x alone by the conversion of u into a function of x.

Since
$$\frac{dv}{du} = \frac{1}{U}, \quad \frac{du}{dx} = QU,$$

the above equation becomes

$$\left(\frac{dv}{dx}\right) = \frac{dv}{dx} + Q.$$

Now if $u = 0$ give $v = 0$ for all values of x, it will therefore give

$$\left(\frac{dv}{dx}\right) = 0,$$

and further,

$$\frac{dv}{dx} = \frac{d}{dx}\int_0^u \frac{du}{U} = 0,$$

since we are permitted to make $u=0$ before effecting the differentiation with respect to x. Hence the equation reduces to

$$0=Q.$$

And this is not satisfied, since by hypothesis Q does not vanish with u.

Hence if $u=0$ make $\int_0^u \frac{du}{U}=0$, the transformed differential equation will no longer have $u=0$ for a solution.

COR. Assuming $Q=1$, which does not violate the hypothesis respecting Q, and gives

$$U=f(x, u),$$

we see that if

$$\frac{du}{dx}=f(x, u)$$

be satisfied by $u=0$, and if at the same time $u=0$ gives

$$\int_0^u \frac{du}{f(x, u)}=0,$$

the differential equation can be transformed so as to cease to admit of the solution $u=0$.

It is obvious however that it is best to assume Q so as to make the subsequent integration for determining v the simplest possible.

It is manifest that a solution which can thus be made to cease to satisfy the differential equation cannot be a particular primitive. For the complete primitive of the transformed differential equation which it does *not* satisfy is convertible into the complete primitive of the original differential equation which it does satisfy, merely by writing therein for v its expression as a function of x and y. It cannot therefore be a case of the complete primitive in any sense. It must be a singular solution of the envelope species.

The converse proposition still remains to be proved.

10. PROPOSITION. *If $u=0$ be a singular solution of the envelope species of a differential equation of the first order, and if by assuming u and x as the variables, the differential equation is reduced to the form*

$$\frac{du}{dx}=f(x, u),$$

then will

$$\int_0^u \frac{du}{f(x, u)}$$

become 0 when $u=0$.

Let the complete primitive be represented by

$$F(x, u) = C,$$

then, since

$$\frac{dF(x, u)}{dx}+\frac{dF(x, u)}{du}\frac{du}{dx}=0,$$

we have if for brevity we represent $F(x, u)$ by F,

$$\frac{du}{dx}=-\frac{\frac{dF}{dx}}{\frac{dF}{du}};$$

therefore
$$\int_0^u \frac{du}{f(x, u)}=-\int_0^u \frac{\frac{dF}{du}}{\frac{dF}{dx}}\,du.$$

Now $u=0$ being a singular solution, $F(x, 0)$ is not a constant; for if it were, the complete primitive would, on giving to C the constant value in question, yield $u=0$ as a particular primitive. And this would equally be the case whether that constant were finite or infinite in value. We see then that $F(x, 0)$ must be a function of x, and therefore $\frac{dF(x, 0)}{dx}$ must either be a function of x, or a finite constant differing from 0; the latter if $F(x, 0)$ be of the form $ax+b$, the former if it be not of that form. Therefore the value of $\frac{dF(x, u)}{dx}$ when $u = 0$, since in this we are permitted to make $u = 0$ before differentiating with respect to x, will be a function of x, or a finite constant differing from 0.

Now it is manifest that in general

$$\int_0^u \frac{1}{\frac{dF}{dx}} \frac{dF}{du} du = H \int_0^u \frac{dF}{du} du,$$

where H is some value intermediate between the greatest and least values which $\frac{1}{\frac{dF}{dx}}$ assumes within the limits of integration. When these limits are, as in the above case, infinitesimal, we have

$$H = \frac{1}{\frac{dF(x, 0)}{dx}}.$$

Hence
$$\int_0^u \frac{du}{f(x, u)} = \frac{1}{\frac{dF(x, 0)}{dx}} \int_0^u \frac{dF}{du} du$$

$$= \frac{1}{\frac{dF(x, 0)}{dx}} \{F(x, u) - F(x, 0)\}.$$

But we have seen that $\frac{dF(x, 0)}{dx}$ does not vanish. Hence its reciprocal, the first factor of the right-hand member of the above equation, does not become infinite. Again,

$$F(x, u) - F(x, 0)$$

vanishing when $u = 0$, we have

$$\int_0^u \frac{du}{f(x, u)} = 0,$$

when u is made infinitesimal as was to be shewn.

It will be observed that the previous general expression for $\int_0^u \frac{du}{f(x, u)}$ becomes infinite if $u=0$ is a particular integral. For then, $F(x, 0)$ being a constant, $\frac{dF(x,0)}{dx}$ vanishes, while $F(x, u) - F(x, 0)$ does not vanish so long as u differs by however small a quantity from 0.

These propositions form the ground of the following Rule for the discrimination of singular solutions of the envelope species from all others.

11. Rule. The proposed solution being represented by $u=0$, let the differential equation, transformed by making u and x the variables, be

$$\frac{du}{dx} + f(x, u) = 0.$$

Determine as a function of x and u the integral

$$\int_0^u \frac{du}{U},$$

in which U is either equal to $f(x, u)$, or to $f(x, u)$ deprived of any factor which neither vanishes nor becomes infinite when $u=0$. If that integral tend to 0 with u the solution is singular.

Ex. 1. Determine whether $y=0$ is a singular solution or particular integral of the differential equation

$$\frac{dy}{dx} = y(\log y)^2.$$

Here, since $u=y$, no preliminary transformation is needed.

We have $$\int_0^y \frac{dy}{y(\log y)^2} = -\frac{1}{\log y},$$

which tends to 0 with y. Hence the solution is singular.

To verify this we observe that the complete primitive is

$$y = \epsilon^{\frac{1}{c-x}},$$

and this cannot be reduced to $y=0$ by giving any constant value to c.

We have seen in Art. 7 that the test which is founded upon the comparison of differential coefficients does not suffice to characterize the above solution.

Ex. 2. The equation $\frac{dy}{dx}=\frac{y\log y}{x}$ is satisfied by $y=0$. Is this solution singular or particular?

Here also no transformation is required. We have, rejecting the factor $\frac{1}{x}$ which neither vanishes nor becomes infinite when $y=0$,

$$\int_0^y \frac{dy}{y\log y} = \log\log y - \log\log 0$$

$$= \log\frac{\log y}{\log 0},$$

and this being infinite, however small y may be, may properly be said to tend to infinity as y tends to 0. The solution is therefore particular.

It will perhaps appear at first sight as if in the above example we ought to write

$$\log\frac{\log y}{\log 0} = \log 1 = 0$$

when y is made equal to 0. But the course of the demonstration shews that the value of the definite integral must be first obtained on the hypothesis that u (in this case replaced by y) is finite, and then the limiting value which its expression approaches to, as u approaches to 0, be sought. And in this case, since for all finite values of u however small the integral is infinite, its limiting value is infinite.

The complete primitive in the above case is

$$y=\epsilon^{cx},$$

and the nature of the solution $y=0$ has already been discussed in Art. 4.

History of the Theory of Singular Solutions.

12. It is remarkable that while the theory of enveloping curves and surfaces was at once founded and developed by Leibnitz in 1692—4*, the corresponding theory of the singular solutions of differential equations has been of very slow growth. The existence of these solutions was first recognised in 1715 by Brook Taylor; it was scarcely more than recognised by Clairaut in 1734. Euler, in a special memoir, entitled *Exposition de quelques Paradoxes dans le Calcul Integral*, published in the Memoirs of the Academy of Berlin for 1756, first made them a direct object of investigation; but the foundations of their true theory were only laid in 1768 in his *Institutiones Calculi Integralis*. Laplace, Lagrange, Legendre, Poisson, Cauchy, and De Morgan have in various ways developed and extended that theory; but there has been so remarkable a want of unity and connexion in this long series of researches, that important portions of the theory appearing in a too isolated form have been neglected, forgotten, and rediscovered. I purpose here to give a brief account of what seems most characteristic, rather than of what is most original in their several researches; for the germs of nearly all subsequent discoveries on the subject are to be found in the great work of Euler.

Taylor and Clairaut appear to have been led by accident to the noticing of singular solutions; the former while directly occupied on the solution of differential equations, the latter while discussing a remarkable class of problems relating to the connecting properties of different branches of the same curve. Taylor gave them the name singular, while Clairaut, and Euler too in his memoir, regarded them as a species of paradox, not merely from their non-inclusion in the general integral, but from the mode of their discovery through a process of differentiation. The memoir of Euler, though it sheds no light on the real nature of these solutions, contains

* *Acta Eruditorum*, 1692, p. 168; 1694, p. 311. *Opera*, Tom. III. pp. 264, 296.
Methodus Incrementorum, p. 26.
Mémoires de l'Académie des Sciences, 1734, p. 209.

an interesting theorem concerning their connexion with the form of the differential equation, viz. If this equation can be brought to the form

$$Vdz = Z(Pdx + Qdy),$$

in which z is a function of x and y, and Z of z, then will

$$Z = 0$$

be a singular solution. In his *Institutiones Calculi Integralis*, Tom. I. p. 393, however, Euler gives a rule which is the counterpart of that of Cauchy. [See Chap. VIII. Art. 12.] He shews that if $u = 0$ be a particular integral, and if the differential equation be reduced to the form

$$\frac{du}{dx} = \phi(x, u),$$

then
$$\int_0^u \frac{du}{\phi(x, u)} = \infty.$$

The limits of integration are here supplied. The reasoning, which is not fully developed, is the following. From the transformed equation we have

$$dx = \frac{du}{\phi(x, u)}.$$

Hence
$$x = C + \int \frac{du}{\phi(x, u)},$$

$$\frac{x}{C} = 1 + \frac{1}{C}\int \frac{du}{\phi(x, u)}.$$

If this be satisfied by a solution involving x and y, and if that solution be a particular integral, then on putting for x its value in terms of u and integrating, the above equation will be satisfied by giving some particular constant value to C. But if the supposed particular integral be $u = 0$, then x and u being independent, we may perform the integration with respect to u as if x were constant. The resulting equation cannot be free from x unless C be infinite, and then it will evidently not be satisfied unless $\int \frac{du}{\phi(x, u)}$ be infinite. We infer then that this is a necessary condition in order that $u = 0$ may be a particular integral.

This is Euler's fundamental theorem, and from this, by means of an hypothesis agreeing with that of Poisson concerning the form of the transformed differential equation, he arrives at the condition

$$\frac{dp}{dy} = \infty.$$

[In the passage to which Professor Boole refers, Euler does not undertake to discuss the nature of *any* solution, but only of a solution of the form $x =$ constant. On his page 408 Euler proceeds to discuss the nature of *any* solution. Professor Boole seems to me to attribute too much to Euler. For the convenience of those who wish to examine the original, I will give the reference to the passages in the later editions of Euler's *Institutiones Calculi Integralis:* Vol. I. pages 343 and 355 of the edition of 1792; Vol. I. pages 342 and 354 of the edition of 1824.]

Laplace in the Memoirs of the French Academy for 1772, p. 343, established the tests

$$\frac{dp}{dy} = \infty, \quad \frac{d}{dx}\left(\frac{1}{p}\right) = \infty,$$

and shewed their respective uses. He established also the test which consists in the comparison of differential coefficients, and he supposes it universal. He adopts the hypothesis of his predecessors as to the forms of expansion, but with some recognition of its insufficiency.

Lagrange in the Memoirs of the Academy of Berlin for 1774, p. 197, and 1779, p. 121, appears first to have developed the theory of singular solutions in its two forms of derivation from the complete primitive and derivation from the differential equation, and to have established the essential connexion of these. But supposing the differential equation to be expressible in the rational form

$$F(x, y, p) = 0,$$

and employing the differential coefficients of $F(x, y, p)$ instead of those of p he was led to sacrifice rigour to symmetry. One of his results has often since been adopted as a test of singular solutions. It may be thus stated.

PROP. A singular solution makes the *general* value of $\frac{d^2y}{dx^2}$, deduced from the differential equation in its rational and integral expression, to assume the form $\frac{0}{0}$.

[The demonstration is given in Chap. VIII. Art. 14.]

This ambiguity of value of $\frac{d^2y}{dx^2}$ is evidently but an expression of the fact that the contact of a curve of the complete primitive and that of the singular solution is not in general of the second order.

The result given in equation (5) of Chap. VIII. Art. 14 has also been adopted as the test of singular solutions.

The researches of Poisson and Cauchy have already been noticed. It is certainly remarkable that the final test to which Cauchy's analysis led should be essentially the same as that which had been discovered by Euler so long before.

Professor De Morgan has thrown an important light upon the nature of the conditions

$$\frac{dp}{dy} = \infty, \quad \frac{dp}{dx} = \infty,$$

which are fulfilled by all singular solutions in the expression of which x and y are both involved. He has shewn that any relation between x and y which satisfies these conditions will satisfy the differential equation unless it make $\frac{d^2y}{dx^2}$, as derived from the differential equation, infinite; that it *may* satisfy the differential equation even if it make $\frac{d^2y}{dx^2}$ infinite; lastly, that if it do not satisfy the differential equation, the curve it represents is a locus of points of infinite curvature, usually cusps, in the curves of complete primitives.

The proof is as follows:

Let $$p = \phi(x, y)$$

be the differential equation. Then the proposed conditions are

$$\frac{d\phi(x, y)}{dy} = \infty, \quad \frac{d\phi(x, y)}{dx} = \infty,$$

therefore by differentiation,

$$\frac{d^2\phi}{dx\,dy} + \frac{d^2\phi}{dy^2}\frac{dy}{dx} = 0, \quad \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dx\,dy}\frac{dy}{dx} = 0,$$

whence we have

$$\frac{dy}{dx} = -\frac{\dfrac{d^2\phi}{dx\,dy}}{\dfrac{d^2\phi}{dy^2}} = -\frac{\dfrac{d^2\phi}{dx^2}}{\dfrac{d^2\phi}{dx\,dy}}.$$

These are two equivalent expressions for the same value of $\frac{dy}{dx}$. The question now is, under what circumstances this value of $\frac{dy}{dx}$ will satisfy the differential equation.

Now from that equation we have by differentiation

$$\frac{d^2y}{dx^2} = \frac{d\phi}{dx} + \frac{d\phi}{dy}\frac{dy}{dx},$$

whence

$$\frac{dy}{dx} = \frac{\dfrac{d^2y}{dx^2} - \dfrac{d\phi}{dx}}{\dfrac{d\phi}{dy}}.$$

If then $\frac{d^2y}{dx^2}$ be finite we have, since $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ are both infinite,

$$\frac{dy}{dx} = -\frac{\frac{d\phi}{dx}}{\frac{d\phi}{dy}},$$

and this by the rule for the evaluation of fractions of the form $\frac{\infty}{\infty}$ is equivalent to the value in either of its forms before obtained for $\frac{dy}{dx}$. Hence, any relation which satisfies the given conditions and makes $\frac{d^2y}{dx^2}$ finite, will satisfy the differential equation.

And the same result holds even if $\frac{d^2y}{dx^2}$ be infinite, provided that $\frac{d^2y}{dx^2} \div \frac{d\phi}{dy}$ vanish.

Lastly, as when this result does not hold, the failure is due to the infinite value of $\frac{d^2y}{dx^2}$, we see that the line in which the locus of the proposed relation intersects the curves of primitives will be a locus of their points of infinite curvature.

[Transactions of the *Cambridge Philosophical Society*, Vol. IX. Part II.]

Legendre's Memoir of 1790 throws but little light upon the subject of this Chapter. But it exhibits the theory of the singular solutions of differential equations of the higher orders, both ordinary and partial, in a form of great beauty, and will be noticed in the proper places.

CHAPTER XXII.

ADDITIONS TO CHAPTER IX.

1. By successive application of the second theorem of Chap. IX. Art. 13, a linear equation of the n^{th} order may be reduced to one of the $(n-r)^{\text{th}}$ order, if r distinct integrals of what the given equation deprived of its second term would be are known.

The reduction may however be effected immediately by the method of the variation of parameters. In this and in most general investigations connected with differential equations great advantages in point of brevity and of the power of expression are gained by the employment of the symbol of summation Σ, and of the language of determinants. I shall exemplify this here.

Suppose the given equation to be

$$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} \dots\dots + A_n = X \dots\dots\dots (1),$$

and let $y_1, y_2, \dots y_r$ be r particular values of y, satisfying the equation

$$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} \dots\dots + A_n = 0 \dots\dots\dots (2).$$

Thus $$y = c_1 y_1 + c_2 y_2 \dots + c_r y_r$$

is a solution of the latter equation including these particular solutions. We shall represent this by

$$y = \Sigma c_i y_i \dots\dots\dots\dots\dots\dots\dots\dots (3),$$

and regarding the quantities $c_1, c_2, \dots c_r$, represented here by

c_i as variable parameters, shall seek to determine them so that the above value of y may satisfy the equation given.

These r parameters, enabling us to satisfy $r-1$ arbitrary conditions, besides satisfying the differential equation, we may choose these so that

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^{r-1}y}{dx^{r-1}}$$

may be the same *in form* as if $c_1, c_2, \ldots c_r$ were constant. Now from (3)

$$\frac{dy}{dx} = \Sigma c_i \frac{dy_i}{dx} + \Sigma y_i \frac{dc_i}{dx},$$

whence

$$\frac{dy}{dx} = \Sigma c_i \frac{dy_i}{dx},$$

provided that the condition

$$\Sigma y_i \frac{dc_i}{dx} = 0$$

be satisfied. Differentiating the first of these equations, we find in the same way that

$$\frac{d^2y}{dx^2} = \Sigma c_i \frac{d^2y_i}{dx^2},$$

provided that the condition

$$\Sigma \frac{dy_i}{dx}\frac{dc_i}{dx} = 0$$

be satisfied. And thus continuing we see that the system of r equations

$$y = \Sigma c_i y_i, \quad \frac{dy}{dx} = \Sigma c_i \frac{dy_i}{dx}, \ldots\ldots \frac{d^{r-1}y}{dx^{r-1}} = \Sigma c_i \frac{d^{r-1}y_i}{dx^{r-1}} \ldots\ldots (4),$$

will hold true provided that the $r-1$ conditions

$$\Sigma y_i \frac{dc_i}{dx} = 0, \quad \Sigma \frac{dy_i}{dx}\frac{dc_i}{dx} = 0, \ldots \Sigma \frac{d^{r-2}y_i}{dx^{r-2}}\frac{dc_i}{dx} = 0 \ldots\ldots (5),$$

be satisfied. In each of these equations the symbol Σ is to be interpreted by giving to i the successive values 1, 2,... r, and taking the sum of the results.

Differentiating the last of the equations (4), we have

$$\frac{d^r y}{dx^r} = \Sigma c_i \frac{d^r y_i}{dx^r} + \Sigma \frac{d^{r-1} y_i}{dx^{r-1}} \frac{dc_i}{dx}.$$

As we cannot impose the condition that the last term of this equation shall vanish, let z represent its unknown value, then

$$\frac{d^r y}{dx^r} = \Sigma c_i \frac{d^r y_i}{dx^r} + z \quad\text{..................... (6).}$$

Now the system of equations (5), together with

$$\Sigma \frac{d^{r-1} y_i}{dx^{r-1}} \frac{dc_i}{dx} = z,$$

constitute a system of r simple algebraic equations determining by solution the r quantities

$$\frac{dc_1}{dx}, \frac{dc_2}{dx}, \ldots \frac{dc_r}{dx}$$

in terms of their coefficients and of z, and therefore in terms of x and z, since the coefficients are known as functions of x. It is evident also that as the second members of all the equations but one vanish, and the second member of that is z, the values so determined will be of the form

$$\frac{dc_1}{dx} = X_1 z, \quad \frac{dc_2}{dx} = X_2 z, \ldots \frac{dc_r}{dx} = X_r z,$$

$X_1, X_2, \ldots X_r$ being known functions of x. Thus the r unknown quantities $\frac{dc_1}{dx}, \ldots \frac{dc_r}{dx}$ are made to depend upon only one unknown quantity, viz. z. It remains then to determine z.

For this purpose we must complete the expression of the differential coefficients of y, and substitute in the given differential equation, and then seek to satisfy that equation.

Now differentiating (6) we have

$$\frac{d^{r+1}y}{dx^{r+1}} = \Sigma c_i \frac{d^{r+1}y_i}{dx^{r+1}} + \Sigma \frac{d^r y_i}{dx^r}\frac{dc_i}{dx} + \frac{dz}{dx}$$

$$= \Sigma c_i \frac{d^{r+1}y_i}{dx^{r+1}} + \Sigma \left(\frac{d^r y_i}{dx^r} X_i\right) z + \frac{dz}{dx}$$

on substituting for $\frac{dc_i}{dx}$ the value $X_i z$ as above determined. We observe that the coefficient of z is here a known function of x. If we differentiate this equation and in the result substitute as above for $\frac{dc_i}{dx}$, we shall have a result of the form

$$\frac{d^{r+2}y}{dx^{r+2}} = \Sigma c_i \frac{d^{r+2}y_i}{dx^{r+2}} + Lz + M\frac{dz}{dx} + \frac{d^2z}{dx^2},$$

L and M being known functions of x. Ultimately then we have

$$\frac{d^n y}{dx^n} = \Sigma c_i \frac{d^n y_i}{dx^n} + Pz + Q\frac{dz}{dx} \ldots\ldots + \frac{d^{n-r}z}{dx^{n-r}}.$$

Thus, while y and the differential coefficients of y up to the $(r-1)^{\text{th}}$ are of the same form as if $c_1, c_2, \ldots c_r$ were constant, the succeeding ones differ in containing an additional portion consisting of z, and differential coefficients of z multiplied by known functions of x. The result of substitution of these values in the given differential equation will therefore consist also of two classes of terms, viz. terms under the sign of summation, which will be the same in form as if $c_1, c_2, \ldots c_r$ were constant, and terms involving the differential coefficients of z up to the $(n-r)^{\text{th}}$, with multipliers which are known functions of x. We shall in fact have

$$\Sigma c_i \left(\frac{d^n y_i}{dx^n} + A_1 \frac{d^{n-1}y_i}{dx^{n-1}} + A_2 \frac{d^{n-2}y_i}{dx^{n-2}} \ldots\ldots + A_n\right)$$

$$+ \frac{d^{n-r}z}{dx^{n-r}} + R\frac{d^{n-r-1}z}{dx^{n-r-1}} \ldots\ldots + S = X.$$

Now y_i being by hypothesis an integral of (2), the first line of the above equation vanishes, and there remains the linear equation of the $(n-r)^{\text{th}}$ order

$$\frac{d^{n-r}z}{dx^{n-r}} + R\frac{d^{n-r-1}z}{dx^{n-r-1}} \dots\dots + S = X.$$

Supposing z hence determined, we have in general

$$c_i = \int X_i z\, dx,$$

and hence

$$y = y_1\int X_1 z\, dx + y_2\int X_2 z\, dx \dots\dots + y_r\int X_r z\, dx,$$

and as z will have $n-r$ distinct values, each involving an arbitrary constant, the above equation will furnish $n-r$ distinct values of y, each involving an arbitrary constant. It is to be observed that no arbitrary constant need be added in the integration of the terms $X_i z\, dx$, for the effect of such addition would only be to reproduce the known integrals $c_i y_i$. In this way, however, the equation would represent the *general* integral of the differential equation given.

2. Let us examine the form of the result in the particular case in which $r = n-1$.

Here we have

$$\frac{d^m y}{dx^m} = \Sigma c_i \frac{d^m y_i}{dx^m}$$

from $m = 0$ to $m = n-2$, then

$$\frac{d^{n-1}y}{dx^{n-1}} = \Sigma c_i \frac{d^{n-1}y_i}{dx^{n-1}} + z,$$

$$\frac{d^n y}{dx^n} = \Sigma c_i \frac{d^n y_i}{dx^n} + \Sigma\left(X_i \frac{d^{n-1}y_i}{dx^{n-1}}\right) z + \frac{dz}{dx}.$$

Accordingly the differential equation for z will be

$$\frac{dz}{dx} + \Sigma\left(X_i \frac{d^{n-1}y_i}{dx^{n-1}}\right) z + A_1 z = X \text{.............} (7).$$

Now the equations for determining

$$\frac{dc_1}{dx}, \frac{dc_2}{dx}, \dots \frac{dc_{n-1}}{dx}$$

become on putting $X_i z$ for $\frac{dc_i}{dx}$, and writing for brevity y'_i for $\frac{dy_i}{dx}$, y_i'' for $\frac{d^2y_i}{dx^2}$, &c.,

$$y_1X_1 + y_2X_2 \dots + y_{n-1}X_{n-1} = 0,$$

$$y'_1X_1 + y'_2X_2 \dots + y'_{n-1}X_{n-1} = 0,$$

....

$$y_1^{(n-3)}X_1 + y_2^{(n-3)}X_2 \dots + y_{n-1}^{(n-3)}X_{n-1} = 0,$$

$$y_1^{(n-2)}X_1 + y_2^{(n-2)}X_2 \dots + y_{n-1}^{(n-2)}X_{n-1} = 1.$$

Whence, by the theory of determinants,

$$X_1 = \frac{1}{M}\frac{dM}{dy_1^{(n-2)}}, \dots\dots X_{n-1} = \frac{1}{M}\frac{dM}{dy_{n-1}^{(n-2)}},$$

M standing for the determinant

$$\begin{vmatrix} y_1, & y_2, \dots\dots & y_{n-1} \\ \dots & \dots & \dots \\ y_1^{(n-2)}, & y_2^{(n-2)}, \dots & y_{n-1}^{(n-2)} \end{vmatrix}$$

Now the determinant is ultimately a function of x; and such indeed that

$$\Sigma\left(X_i \frac{d^{n-1}y_i}{dx^{n-1}}\right) = \frac{1}{M}\frac{dM}{dx}.$$

For

$$\frac{dM}{dx} = \Sigma \frac{dM}{dy_i}\frac{dy_i}{dx} + \Sigma \frac{dM}{dy_i'}\frac{dy_i'}{dx} \ldots + \Sigma \frac{dM}{dy_i^{(n-2)}}\frac{dy_i^{(n-2)}}{dx}$$

$$= \Sigma \frac{dM}{dy_i} y_i' + \Sigma \frac{dM}{dy_i'} y_i'' \ldots + \Sigma \frac{dM}{dy_i^{(n-2)}} y_i^{(n-1)} \;\ldots\ldots\ldots (8).$$

Now M being homogeneous and of the first degree with respect to the quantities $y_1, y_2, \ldots\, y_{n-2}$, we have

$$\Sigma \frac{dM}{dy_i} y_i = M.$$

Hence $\Sigma \dfrac{dM}{dy_i} y_i'$ is what M becomes when in its expression $y_1, y_2, \ldots y_{n-1}$ are changed into $y_1', y_2', \ldots\, y'_{n-1}$, therefore it is what M becomes when two of its rows of elements become identical; therefore it vanishes. In like manner all the other sums in (8) vanish excepting the last, for $y_1^{(n-1)}, \ldots\, y_{n-1}^{(n-1)}$ is not a row of elements of the determinant M. Thus we have

$$\Sigma \frac{dM}{dy_i^{(n-2)}} y_i^{(n-1)} = \frac{dM}{dx}.$$

Hence

$$\Sigma X_i y_i^{(n-1)} = \Sigma \frac{1}{M}\frac{dM}{dy_i^{(n-2)}} y_i^{(n-1)} = \frac{1}{M}\frac{dM}{dx}.$$

Thus the equation (7) becomes

$$\frac{dz}{dx} + \left(\frac{1}{M}\frac{dM}{dx} + A_1\right) z = X;$$

therefore $$z = \frac{1}{M}\epsilon^{-\int A_1 dx}\int M\epsilon^{\int A_1 dx} X dx.$$

Hence, since

$$\frac{dc_i}{dx} = X_i z = \frac{1}{M} \frac{dM}{dy_i^{(n-2)}} z,$$

whence

$$c_i = \int \frac{1}{M} \frac{dM}{dy_i^{(n-2)}} z\,dx,$$

we have

$$y = \Sigma y_i \int \frac{1}{M} \frac{dM}{dy_i^{(n-2)}} z\,dx,$$

z being given above.

In the case of $X = 0$, we have

$$z = \frac{C}{M} \epsilon^{-\int A_1 dx},$$

whence

$$y = \Sigma y_i \int \frac{C}{M^2} \frac{dM}{dy_i^{(n-2)}} \epsilon^{-\int A_1 dx}\,dx.$$

CHAPTER XXIII.

ADDITIONS TO CHAPTER X.

1. THE theory of singular solutions of differential equations of the higher orders has been presented in the most complete form which it has yet received by Legendre. (*Mémoires de l'Académie Royale des Sciences*, 1790, p. 218.) He determines first the possible forms of these solutions considered as emerging from the complete primitive by the variations of its arbitrary constants, and secondly the theory of their derivation from the differential equation itself. I shall follow the same order, and shall in the end endeavour to point out in what respect Legendre's theory may be regarded as complete, and in what respect it is imperfect.

Suppose the differential equation to be of the n^{th} order, and let it when solved with respect to the highest differential coefficient of y be represented by

$$y_n = \phi(x, y, y_1, y_2, \dots y_{n-1}) \quad\quad (1),$$

in which, for brevity,

$$y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2}, \dots y_n = \frac{d^ny}{dx^n}.$$

Let also its complete primitive, solved with respect to y, be represented by

$$y = f(x, a_1, a_2, \dots a_n) \quad\quad (2),$$

$a_1, a_2, \dots a_n$ being the arbitrary constants of the solution. If we differentiate (2) with respect to x, regarding $a_1, a_2, \dots a_n$ no longer as constants but as functions of x, so to be determined as to leave the expressions for $y_1, y_2, \dots y_n$ as functions of $a_1, a_2, \ldots a_n$ the same as before, we shall have, on representing the second member of (2) by f,

$$y_1 = \frac{df}{dx} + \frac{df}{da_1}\frac{da_1}{dx} + \frac{df}{da_2}\frac{da_2}{dx} \ldots\ldots + \frac{df}{da_n}\frac{da_n}{dx},$$

whence

$$y_1 = \frac{df}{dx},$$

provided that

$$\frac{df}{da_1}\frac{da_1}{dx} + \frac{df}{da_2}\frac{da_2}{dx} \ldots\ldots + \frac{df}{da_n}\frac{da_n}{dx} = 0.$$

Differentiating on the same hypothesis the first of these two equations, we find in the same way

$$y_2 = \frac{d^2 f}{dx^2},$$

provided that

$$\frac{d^2 f}{dx\,da_1}\frac{da_1}{dx} + \frac{d^2 f}{dx\,da_2}\frac{da_2}{dx} \ldots\ldots + \frac{d^2 f}{dx\,da_n}\frac{da_n}{dx} = 0.$$

And continuing thus, it results that the system

$$y_1 = \frac{df}{dx}, \quad y_2 = \frac{d^2 f}{dx^2}, \ldots y_n = \frac{d^n f}{dx^n} \ldots\ldots\ldots\ldots (3),$$

will be satisfied, i.e., that $y_1, y_2, \ldots y_n$ will have the same expressions when $a_1, a_2, \ldots a_n$ are variable as they have when these are constant, provided that the law of their variation be determined by the conditions

$$\left.\begin{aligned} \frac{df}{da_1}\frac{da_1}{dx} + \frac{df}{da_2}\frac{da_2}{dx} \ldots\ldots + \frac{df}{da_n}\frac{da_n}{dx} &= 0 \\ \frac{d^2 f}{da_1\,dx}\frac{da_1}{dx} + \frac{d^2 f}{da_2\,dx}\frac{da_2}{dx} \ldots\ldots + \frac{d^2 f}{da_n\,dx}\frac{da_n}{dx} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \frac{d^n f}{da_1\,dx^{n-1}}\frac{da_1}{dx} + \frac{d^n f}{da_2\,dx^{n-1}}\frac{da_2}{dx} \ldots\ldots + \frac{d^n f}{da_n\,dx^{n-1}}\frac{da_n}{dx} &= 0 \end{aligned}\right\} \ldots (4).$$

In this system the coefficients of

$$\frac{da_1}{dx}, \frac{da_2}{dx}, \dots \frac{da_n}{dx}$$

are known functions of x, a_1, a_2, ... a_n when the form of f is known.

Eliminating

$$\frac{da_1}{dx}, \frac{da_2}{dx}, \dots \frac{da_n}{dx},$$

we have a relation between x, a_1, a_2, ... a_n; and this relation, with the given complete primitive and the first $n-1$ of the derived and reduced equations, viz., with

$$y = f, \quad y_1 = \frac{df}{dx}, \quad y_2 = \frac{d^2 f}{dx^2}, \dots y_{n-1} = \frac{d^{n-1} f}{dx^{n-1}},$$

will enable us to eliminate a_1, a_2, ... a_n, and to obtain a relation of the form

$$\psi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^{n-1}y}{dx^{n-1}}\right) = 0 \dots\dots\dots (5).$$

This is a differential equation of the $(n-1)^{\text{th}}$ order. It differs in its origin from the given differential equation, in that a *new* relation between x, a_1, a_2, ... a_n has been employed in place of the n^{th} equation, derived by differentiation from the complete primitive, for the elimination of the constants.

The differential equation of the $(n-1)^{\text{th}}$ order thus obtained has an integral expressing y in terms of x, and $n-1$ arbitrary constants. This is the most general form of a singular solution of the differential equation.

It is possible that the elimination of a_1, a_2, ... a_n may lead to a resulting differential equation which, instead of being of the order $n-1$, is of the order $n-2$, $n-3$, &c. The complete integral of such equation would be a singular solution of the differential equation. These possible types of solutions are distinguished by Legendre according to the number of arbitrary constants which they contain. A solution containing $n-1$ arbitrary constants is called by him a singular solution of the first order; one containing $n-2$ arbitrary constants a singular solution of the second order, and so on.

Adopting this language we might term the complete primitive a singular solution of the order 0.

Lastly, any relation between x and y, which satisfies the given differential equation, will constitute a particular case, either of the complete primitive or of one of the general forms of singular solutions above defined. In the case of differential equations of the first order it is seen that no arbitrary constant can appear in the expression of the singular solution.

Ex. The equation

$$y - x\frac{dy}{dx} + \frac{1}{2}x^2\frac{d^2y}{dx^2} - \left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)^2 = 0$$

has for its complete primitive

$$y = \frac{ax^2}{2} + bx + a^2 + b^2 \qquad (6),$$

required its singular solution.

Proceeding as above, we find on the hypothesis of a and b being variable parameters, the same formal expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ as if those parameters were constant, viz.

$$\left.\begin{aligned} \frac{dy}{dx} &= ax + b \\ \frac{d^2y}{dx^2} &= a \end{aligned}\right\} \qquad (7),$$

provided that the variation of a and b be such as to satisfy the conditions

$$\left.\begin{aligned} \left(\frac{x^2}{2} + 2a\right)\frac{da}{dx} + (x + 2b)\frac{db}{dx} &= 0 \\ x\frac{da}{dx} + \frac{db}{dx} &= 0 \end{aligned}\right\} \qquad (8).$$

Eliminating hence $\frac{da}{dx}$ and $\frac{db}{dx}$, we have

$$2a - \frac{x^2}{2} - 2bx = 0.$$

And from this, the complete primitive, and the first of the derived equations (7) eliminating a and b, we find

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{x^3}{2} + x\right)\frac{dy}{dx} - (1 + x^2)\,y - \frac{x^4}{16} = 0 \text{ ......(9).}$$

This is the differential equation of the first order, by the solution of which the most general form of the singular solutions of the given differential equation will be determined.

Reducing it to the form

$$\frac{4dy + (2x + x^3)\,dx}{(16y + 4x^2 + x^4)^{\frac{1}{2}}} = (1 + x^2)^{\frac{1}{2}}\,dx,$$

and integrating, we find

$$(16y + 4x^2 + x^4)^{\frac{1}{2}} = x\,(1 + x^2)^{\frac{1}{2}} + \log\{x + \sqrt{(1 + x^2)}\} + C.$$

This then is the general expression for the singular solutions of the given differential equation. We see that it involves in its expression *one* arbitrary constant.

The differential equation (9) may properly be termed a singular first integral of the given differential equation. The singular first integral (9) has itself also a singular solution, viz.

$$y = -\frac{1}{4}x^2 - \frac{1}{16}x^4;$$

but this is not a solution of the original differential equation. Nor have we any right to expect that it should be so. A singular solution of a differential equation of the first order does not necessarily satisfy the differential equations of higher orders derived from that equation, Chapter XXII. Art. 7.

2. It remains to establish the theory of the derivation of the singular solution from the differential equation without the mediation of the complete primitive.

Resuming the differential equation in its reduced form (1), and representing its second member by ϕ, suppose an infinitesimal variation given to the arbitrary constants of its complete primitive, and let the symbol δ be used to denote the corresponding derived variations of $y, y_1, \ldots y_n$. Then we have

$$\delta y_n = \frac{d\phi}{dy}\delta y + \frac{d\phi}{dy_1}\delta y_1 \ldots + \frac{d\phi}{dy_{n-1}}\delta y_{n-1}.$$

But $\delta y_1 = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}, \quad \delta y_2 = \frac{\delta d^2 y}{dx^2} = \frac{d^2\delta y}{dx^2},$

and so on. Hence, substituting and transposing,

$$\frac{d^n\delta y}{dx^n} - \frac{d\phi}{dy_{n-1}}\frac{d^{n-1}\delta y}{dx^{n-1}} - \frac{d\phi}{dy_{n-2}}\frac{d^{n-2}\delta y}{dx^{n-2}} - \text{\&c.} = 0 \ldots\ldots (10).$$

Let us consider the real nature of this equation.

If a value of y, suppose $y = \psi(x)$, satisfy the given differential equation, that value substituted in the coefficients

$$\frac{d\phi}{dy_{n-1}}, \quad \frac{d\phi}{dy_{n-2}}, \text{ \&c.}$$

of the above equation will convert them into functions of x, and the equation itself will become a linear differential equation, the solution of which will determine δy as a function of x. If the differential equation (10) be really, as it is apparently, of the n^{th} degree, δy will have n arbitrary constants, $a_1, \ldots a_n$, and will be of the form

$$\delta y = a_1 P_1 + a_2 P_2 \ldots + a_n P_n,$$

$P_1, P_2, \ldots P_n$ being functions of x. Hence

$$y + \delta y = \psi(x) + a_1 P_1 \ldots + a_n P_n.$$

We see thus that the given solution $y = \psi(x)$ will be a *particular* case of this general integral involving n constants. It will therefore be a particular integral of the proposed.

If, owing to the constitution of its coefficients, the differential equation (10) be of the degree $n-1$, we shall have

$$y+\delta y=\psi(x)+\alpha_1 P_1 \ldots+\alpha_{n-1} P_{n-1},$$

and $y=\psi(x)$ will then be a particular case of a solution involving $n-1$ arbitrary constants. It will therefore be a singular solution of the first order. Even so, if the differential equation (10) be of the degree $n-2$, $y=\psi(x)$ will be a singular solution of the second order. And generally, if the differential equation be of the r^{th} degree, $y=\psi(x)$ will be a singular solution of the order $n-r$.

Resuming the equation (10) it is evident that it cannot be of the degree $n-1$, unless $\dfrac{d\phi}{dy_{n-1}}$ be infinite. For, dividing by $\dfrac{d\phi}{dy_{n-1}}$, we have

$$\frac{1}{\dfrac{d\phi}{dy_{n-1}}}\cdot\frac{d^n\delta y}{dx^n}-\frac{d^{n-1}\delta y}{dx^{n-1}}-\&c.=0,$$

in which the first term does not vanish unless $\dfrac{d\phi}{dy_{n-1}}$ be infinite. This then is the necessary condition for a singular solution of the first order. For one of the second order we must have in like manner

$$\frac{d\phi}{dy_{n-1}}=\infty,\quad \frac{d\phi}{dy_{n-2}}=\infty;$$

and so on.

It follows hence that to find the singular solutions of a differential equation of the n^{th} order, we ought to differentiate the equation, regarding y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, &c. as varying through the variation of the arbitrary constants, to form in this way a linear differential equation for δy, to examine the conditions under which this equation reduces to the $(n-1)^{\text{th}}$, or to a lower degree, and to examine whether the most general relation between x and y which satisfies such condition, satisfies also the given differential equation. If so it may be regarded as a singular solution.

Resuming the last Example, viz.

$$y - x\frac{dy}{dx} + \frac{1}{2}x^2\frac{d^2y}{dx^2} - \left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)^2 = 0,$$

and operating with δ we have

$$\left\{\frac{1}{2}x^2 - 2\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)\right\}\frac{d^2\delta y}{dx^2}$$

$$-\left\{x + 2\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)\right\}\frac{d\delta y}{dx} + \delta y = 0,$$

which reduces to a linear differential equation of the first order for determining δy, provided that we have

$$\frac{1}{2}x^2 - 2\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right) = 0.$$

Eliminating $\frac{d^2y}{dx^2}$ from the given equation by means of this there results

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{x^3}{2} + x\right)\frac{dy}{dx} - (1 + x^2)\,y - \frac{x^4}{16} = 0,$$

and we find on differentiating this that it does constitute a solution of the given equation. It is therefore a *singular* first integral of that equation. We see that it agrees with the result obtained under the same name in the previous Article, and the rest of the solution need not be repeated.

3. Upon Legendre's theory, and upon its results, the following observations may be made.

1st. We learn from it that there may exist different *general* forms of the solution of a differential equation of the n^{th} order, viz. the complete primitive involving n arbitrary constants, and general forms of singular solutions containing fewer than n arbitrary constants. A solution $y=\psi(x)$ of unknown origin being given, we construct a differential equation for determining δy, and, solving it, form the expression for $y+\delta y$, and from the number of infinitesimal arbitrary constants it contains, determine the nature of that general value of y of which the given value is a particular case. Now we are not to infer from this that the form of $y+\delta y$ will be the same as the general value of y in question. But we may infer that it will be a form to which that general value is reducible. And the actual reduction will be effected by expressing the general solution (as is always possible) in a form permitting its expansion in ascending powers of the arbitrary constants, and in the expansion making these constants infinitesimal, and rejecting all powers of them above the first. In fact, if

$$y=f(x, a_1, a_2, \dots a_r)$$

be any *general* form of solution which, when we assign to $a_1, a_2, \dots a_r$ particular values (e.g. make them vanish) reduces to

$$y=\psi(x),$$

then we shall have

$$y+\delta y=\psi(x)+\left(\frac{df}{da_1}\right)\delta a_1+\left(\frac{df}{da_2}\right)\delta a_2 \dots +\left(\frac{df}{da_r}\right)\delta a_r,$$

the brackets denoting that after differentiation we make $a_1, a_2, \dots a_r$ vanish.

This is that limiting form of the solution which Legendre's method enables us to construct by the solution of a linear differential equation; and the ground of the sufficiency of his method consists in this, that the infinitesimal quantities

$$\delta a_1, \delta a_2, \dots \delta a_r,$$

which are in fact the arbitrary constants of that solution, are equal in number to the arbitrary constants of the general unlimited solution, the nature of which is thus made known.

2ndly. Legendre's tests for differential equations of the higher orders are in kind and effect analogous to the tests

$$\frac{dp}{dy} = \infty, \quad \frac{d}{dx}\frac{1}{p} = \infty$$

for differential equations of the first order. They enable us to decide whether a solution possesses singularity, not whether it possesses the envelope species of singularity. The completion of Legendre's theory would consist in the discovery of those further tests dependent upon integration which correspond to the test of Euler and Cauchy for differential equations of the first order.

CHAPTER XXIV.

ADDITIONS TO CHAPTER XIV.

[Art. 1 was intended to follow Chap. XIV. Art. 2.]

1. As the condition of dependence of functions of two variables is of fundamental importance in connexion with the theory of ordinary differential equations, so the generalized condition of dependence of functions of any number of variables forms a fundamental part of the theory of partial differential equations. This is contained in the following proposition.

PROP. I. If $u_1, u_2, \ldots u_n$ are functions of $x_1, x_2, \ldots x_n$, but are as such so related that some one of them is expressible as a function of the others, or more generally that there exists among them some identical equation of the form

$$F(u_1, u_2, \ldots u_n) = 0, \qquad (1),$$

so that as functions of $x_1, x_2, \ldots x_n$ they are not mutually independent, then, adopting the notation of determinants, the condition

$$\begin{vmatrix} \frac{du_1}{dx_1}, & \frac{du_1}{dx_2}, & \ldots\ldots & \frac{du_1}{dx_n} \\ \frac{du_2}{dx_1}, & \frac{du_2}{dx_2}, & \ldots\ldots & \frac{du_2}{dx_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{du_n}{dx_1}, & \frac{du_n}{dx_2}, & \ldots\ldots & \frac{du_n}{dx_n} \end{vmatrix} = 0 \qquad (2),$$

is identically satisfied. Conversely, if the above condition be identically satisfied, the functions $u_1, u_2, \ldots u_n$ are not mutually independent in the sense above explained.

First let it be noticed that the Proposition is but a generalization of that of Chap. II. Supposing U and u to be two functions of x and y, the condition of their dependence is affirmed to be

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy} \\ \frac{du}{dx}, & \frac{du}{dy} \end{vmatrix} = 0,$$

i. e. it is the result of eliminating dx, dy, from the equations

$$\frac{dU}{dx}dx + \frac{dU}{dy}dy = 0,$$

$$\frac{du}{dx}dx + \frac{du}{dy}dy = 0,$$

and therefore it is

$$\frac{dU}{dx}\frac{du}{dy} - \frac{dU}{dy}\frac{du}{dx} = 0,$$

as expressed in Chap. II.

We proceed to the general demonstration.

Let the first member of (1), considered as a function of $u_1, u_2, \dots u_n$ be represented for brevity by F; then differentiating, we have

$$\frac{dF}{du_1}du_1 + \frac{dF}{du_2}du_2 + \dots + \frac{dF}{du_n}du_n = 0,$$

from which it follows that if $du_1, du_2, \dots du_{n-1}$ are equal to 0, then is du_n equal to 0; or, since $u_1, u_2, \dots u_n$ are functions of $x_1, x_2, \dots x_n$, that if

$$\left.\begin{aligned} \frac{du_1}{dx_1}dx_1 + \frac{du_1}{dx_2}dx_2 \dots + \frac{du_1}{dx_n}dx_n = 0 \\ \frac{du_2}{dx_1}dx_1 + \frac{du_2}{dx_2}dx_2 \dots + \frac{du_2}{dx_n}dx_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{du_{n-1}}{dx_1}dx_1 + \frac{du_{n-1}}{dx_2}dx_2 \dots + \frac{du_{n-1}}{dx_n}dx_n = 0 \end{aligned}\right\} \dots\dots\dots\dots (3),$$

then is

$$\frac{du_n}{dx_1}dx_1+\frac{du_n}{dx_2}dx_2\ldots+\frac{du_n}{dx_n}dx_n=0\ldots\ldots\ldots\ldots\ldots(4).$$

Thus the last n equations, linear with respect to

$$dx_1,\ dx_2,\ \ldots dx_n,$$

are not independent, and therefore by the theory of linear equations the determinant of the system vanishes identically. Now this is expressed by the condition (2).

It remains to prove the converse, viz. that if the condition (2) be identically satisfied, the functions $u_1, u_2, \ldots u_n$ will not be mutually independent.

First, the $n-1$ functions $u_1, u_2, \ldots u_{n-1}$ are either mutually independent or not mutually independent.

If not, then the n functions $u_1, u_2, \ldots u_n$ are not mutually independent, and the Proposition to be proved is granted.

If $u_1, u_2, \ldots u_{n-1}$ are mutually independent as functions of $x_1, x_2, \ldots x_n$, they may be made to take the place of $n-1$ of these quantities, e.g. $x_1, x_2, \ldots x_{n-1}$ in the expressing of u_n, i.e. we may, by means of the expressions for $u_1, u_2, \ldots u_{n-1}$, eliminate from that of u_n the quantities $x_1, x_2, \ldots x_{n-1}$, and so express u_n as a function of $u_1, u_2, \ldots u_{n-1}$ and x_n. Suppose this done, then the system (3), (4) will be converted into

$$du_1=0,\quad du_2=0,\ \ldots\ldots\ du_{n-1}=0,$$

$$\frac{du_n}{du_1}du_1+\frac{du_n}{du_2}du_2\ldots+\frac{du_n}{du_{n-1}}du_{n-1}+\frac{du_n}{dx_n}dx_n=0.$$

Now, the determinant (2) vanishing, the equations of the linear system (3), (4) are not independent; therefore those of the transformed system, as written above, are not independent; therefore the last equation of that system must be a consequence of the others which manifestly are independent. But from the form of that last equation we see that such cannot be the case unless we have

$$\frac{du_n}{dx_n}=0,$$

which implies that u_n is a function of $u_1, u_2, \dots u_{n-1}$ merely. Hence the functions $u_1, u_2, \dots u_n$ are not independent, as was to be shewn.

The first member of the equation of condition (2) is commonly called the *functional determinant* of $u_1, u_2, \dots u_n$ with respect to $x_1, x_2, \dots x_n$. The proposition may therefore be expressed as follows.

The condition of dependence or independence of any system of functions of as many variables is the vanishing or non-vanishing of the functional determinant of the system.

On account of the great importance of this proposition it is desirable to illustrate it by an example.

Ex. Are the functions

$$x + 2y + z, \quad x - 2y + 3z, \quad 2xy - xz + 4yz - 2z^2$$

mutually independent or not?

The equation of condition is

$$\begin{vmatrix} 1, & 2, & 1 \\ 1, & -2, & 3 \\ 2y - z, & 2x + 4z, & -x + 4y - 4z \end{vmatrix} = 0,$$

that is,

$$-4(-x + 4y - 4z) + 8(2y - z) - 2(2x + 4z) = 0,$$

which is identically satisfied. Hence the functions are dependent. In fact, representing them by u, v, w, we have

$$4w = u^2 - v^2.$$

[Art. 2 was intended to follow Chap. XIV. Art. 4.]

2. As it has been shewn that a primitive

$$u = \phi(v) \text{ .... } (1)$$

leads to a linear partial differential equation of the form

$$Pp + Qq = R \text{ .... } (2),$$

provided that $u=a$, $v=b$, are integrals of the system of ordinary differential equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}\ \dots\dots\dots\dots(3),$$

it is evident that we shall obtain a solution of the partial differential equation (2) by constructing the system of ordinary differential equations (3), deducing their general integrals

$$u=a, \quad v=b,$$

and then constructing from these the primitive (1).

But the question arises, Will this be the most general solution of the partial differential equation given?

That it will be so, may be shewn by means of the general proposition. See Art. 1.

For let $w=0$ represent any solution whatever of the given partial differential equation. Differentiating this with respect to x and y, we have

$$\frac{dw}{dx}+\frac{dw}{dz}p=0, \quad \frac{dw}{dy}+\frac{dw}{dz}q=0,$$

substituting the values of p and q formed from this in the given equation, we have

$$P\frac{dw}{dx}+Q\frac{dw}{dy}+R\frac{dw}{dz}=0,$$

which must be identically satisfied.

In like manner, $u=a$, $v=b$ being solutions of the same equation, we find

$$P\frac{du}{dx}+Q\frac{du}{dy}+R\frac{du}{dz}=0,$$

$$P\frac{dv}{dx}+Q\frac{dv}{dy}+R\frac{dv}{dz}=0,$$

which must be identically satisfied.

Eliminating P, Q, R from these three equations, it results that *the functional determinant of w, u, v, with respect to x, y, z,* will identically vanish. Hence w is a function of u and v, and the equation $w=0$ is a particular case of

$$F(u, v) = 0,$$

which is thus shewn to be the *general* integral of the given equation.

We are thus led to the following general Rule.

RULE. *To integrate the equation $Pp + Qq = R$ we must form the system of ordinary differential equations*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

deduce their general integrals in the form

$$u = a, \quad v = b,$$

and construct the equation

$$F(u, v) = 0.$$

This will be the general solution sought.

[Art. 3 was intended to follow Chap. XIV. Art. 5.]

3. The above theory may be extended to linear partial differential equations of the first order, without regard to the number of the variables.

First, the theory of the genesis of such equations is expressed in the following proposition.

PROP. A primitive equation of the form

$$F(u_1, u_2, \ldots u_n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

in which $u_1, u_2, \ldots u_n$ are any given functions of the variables z, dependent, and $x_1, x_2, \ldots x_n$ independent, will satisfy the linear partial differential equation obtained by eliminating dz, dx_1, $dx_2, \ldots dx_n$ from

$$du_1 = 0, \; du_2 = 0, \ldots\ldots du_n = 0,$$

expressed as total differential equations with respect to the primitive variables, and the equation

$$dz - p_1 dx_1 - p_2 dx_2 \ldots - p_n dx_n = 0.$$

Of this important proposition I propose to give two distinct proofs.

1st proof. Forming the total differential of the given equation we have, on representing its first member by F,

$$\frac{dF}{du_1} du_1 + \frac{dF}{du_2} du_2 \ldots + \frac{dF}{du_n} du_n = 0.$$

Now this cannot be true for all forms of the function unless we have the separate conditions

$$du_1 = 0, \quad du_2 = 0, \ldots\ldots du_n = 0.$$

Strictly to prove this, suppose $F_1, F_2, \ldots F_n$ to be any n distinct and independent functions of $u_1, u_2, \ldots u_n$, and as such, distinct and independent forms of F. Then the above equation gives

$$\frac{dF_1}{du_1} du_1 + \frac{dF_1}{du_2} du_2 \ldots + \frac{dF_1}{du_n} du_n = 0,$$

$$\frac{dF_2}{du_1} du_1 + \frac{dF_2}{du_2} du_2 \ldots + \frac{dF_2}{du_n} du_n = 0,$$

....

$$\frac{dF_n}{du_1} du_1 + \frac{dF_n}{du_2} du_2 \ldots + \frac{dF_n}{du_n} du_n = 0.$$

Now $F_1, F_2, \ldots F_n$ being independent, their functional determinant with respect to $u_1, u_2, \ldots u_n$, does not vanish. This again is the condition necessary and sufficient that the above system of linear equations may be independent; and this lastly being the case, their only possible solution will be

$$du_1 = 0, \quad du_2 = 0, \ldots\ldots du_n = 0,$$

as was to be shewn.

These equations in their developed expression

$$\frac{du_1}{dx_1}dx_1 + \frac{du_1}{dx_2}dx_2 \dots + \frac{du_1}{dx_n}dx_n + \frac{du_1}{dz}dz = 0,$$

$$\frac{du_2}{dx_1}dx_1 \dots\dots\dots\dots\dots\dots + \frac{du_2}{dz}dz = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$\frac{du_n}{dx_1}dx_1 \dots\dots\dots\dots\dots\dots + \frac{du_n}{dz}dz = 0,$$

enable us to determine the ratios of $dx_1, dx_2, \dots, dx_n, dz$ in the form

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} \dots = \frac{dx_n}{X_n} = \frac{dz}{R} \quad \dots\dots\dots (2),$$

where $X_1, X_2, \dots X_n, R$ are functions of the original variables. And now, forming the equation

$$p_1 dx_1 + p_2 dx_2 \dots + p_n dx_n - dz = 0,$$

and eliminating the differentials, we find

$$X_1 p_1 + X_2 p_2 \dots + X_n p_n = R$$

for the partial differential equation sought.

2nd proof. Differentiating the given primitive with respect to x_1, as contained explicitly in the functions $u_1, u_2, \dots u_n$, and also implicitly in the same through z, we have, on representing the first member of the equation by F,

$$\frac{dF}{du_1}\left(\frac{du_1}{dx_1} + p_1\frac{du_1}{dz}\right) + \frac{dF}{du_2}\left(\frac{du_2}{dx_1} + p_1\frac{du_2}{dz}\right) \dots$$
$$+ \frac{dF}{du_n}\left(\frac{du_n}{dx_1} + p_1\frac{du_n}{dz}\right) = 0,$$

or

$$\frac{dF}{du_1}\frac{du_1}{dx_1} + \frac{dF}{du_2}\frac{du_2}{dx_1} \dots + \frac{dF}{du_n}\frac{du_n}{dx_1} + \frac{dF}{dz}p_1 = 0,$$

since

$$\frac{dF}{du_1}\frac{du_1}{dz} + \frac{dF}{du_2}\frac{du_2}{dz} \dots + \frac{dF}{du_n}\frac{du_n}{dz} = \frac{dF}{dz}.$$

Differentiating thus with respect to the remaining independent variables, we obtain finally the system

$$\frac{dF}{du_1}\frac{du_1}{dx_1}+\frac{dF}{du_2}\frac{du_2}{dx_1}\cdots+\frac{dF}{du_n}\frac{du_n}{dx_1}+\frac{dF}{dz}p_1=0,$$

$$\frac{dF}{du_1}\frac{du_1}{dx_2}+\frac{dF}{du_2}\frac{du_2}{dx_2}\cdots+\frac{dF}{du_n}\frac{du_n}{dx_2}+\frac{dF}{dz}p_2=0,$$

....

$$\frac{dF}{du_1}\frac{du_1}{dx_n}+\frac{dF}{du_2}\frac{du_2}{dx_n}\cdots+\frac{dF}{du_n}\frac{du_n}{dx_n}+\frac{dF}{dz}p_n+0,$$

from which, in combination with the equation

$$\frac{dF}{du_1}\frac{du_1}{dz}+\frac{dF}{du_2}\frac{du_2}{dz}\cdots+\frac{dF}{du_n}\frac{du_n}{dz}-\frac{dF}{dz}=0,$$

we can eliminate

$$\frac{dF}{du_1},\ \frac{dF}{du_2},\cdots\cdots\frac{dF}{du_n},\ \frac{dF}{dz}.$$

The result will be

$$\begin{vmatrix} \frac{du_1}{dx_1}, & \frac{du_2}{dx_1}, \cdots\cdots & \frac{du_n}{dx_1}, & p_1 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{du_1}{dx_n}, & \frac{du_2}{dx_n}, \cdots\cdots & \frac{du_n}{dx_n}, & p_n \\ \frac{du_1}{dz}, & \frac{du_2}{dz}, \cdots\cdots & \frac{du_n}{dz}, & -1 \end{vmatrix} = 0,$$

or, converting rows into columns,

$$\begin{vmatrix} \frac{du_1}{dx_1}, & \frac{du_1}{dx_2}, \cdots\cdots & \frac{du_1}{dx_n}, & \frac{du_1}{dz} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{du_n}{dx_1}, & \frac{du_n}{dx_2}, \cdots\cdots & \frac{du_n}{dx_n}, & \frac{du_n}{dz} \\ p_1, & p_2, \cdots\cdots & p_n, & -1 \end{vmatrix} = 0.$$

which is the determinant form of the result affirmed in the proposition.

The second of the above forms of demonstration seems to be preferable to the first, in that it rests only upon the consideration of the one general form of the function F. I have, however, given the two proofs, chiefly in order to illustrate an important remark, viz. that, in nearly all general researches connected with partial differential equations of the first order, two modes of procedure, the one involving the use of differentials, the other that of differential coefficients, may be employed, and that between the forms to which these respective modes give rise, a certain law of reciprocity will be found to exist.

The theory of the solution of the partial differential equation

$$X_1 p_1 + X_2 p_2 \ldots + X_n p_n = R$$

follows immediately from that of its genesis. If we represent by

$$u_1 = a_1, \quad u_2 = a_2, \ldots u_n = a_n,$$

the integrals of the system of ordinary differential equations (2) a solution of the given partial differential equation will be represented by (1). That this will be also the most general solution may be shewn by the argument of Art. 1. For if $w = 0$ represent any solution, then since

$$\frac{dw}{dx_1} + p_1 \frac{dw}{dz} = 0, \ldots\ldots \frac{dw}{dx_n} + p_n \frac{dw}{dz} = 0,$$

we find

$$X_1 \frac{dw}{dx_1} + X_2 \frac{dw}{dx_2} \ldots + X_n \frac{dw}{dx_n} + R \frac{dw}{dz} = 0,$$

from which, in combination with the corresponding equations,

$$X_1 \frac{du_1}{dx_1} + X_2 \frac{du_1}{dx_2} \ldots + X_n \frac{du_1}{dx_n} + R \frac{du_1}{dz} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$X_1 \frac{du_n}{dx_1} + X_2 \frac{du_n}{dx_2} \ldots + X_n \frac{du_n}{dx_n} + R \frac{du_n}{dz} = 0,$$

eliminating $X_1, X_2, \ldots X_n, R$ we obtain a result which expresses that the functional determinant of $w, u_1, \ldots u_n$ with respect to the original variables is virtually 0. Whence w is a function of $u_1, u_2, \ldots u_n$, and the proposed solution is included in the one to which the above method of solution leads.

That method may therefore be stated in the following Rule.

RULE. *To integrate the linear partial differential equation*

$$X_1 \frac{dz}{dx_1} + X_2 \frac{dz}{dx_2} \ldots + X_n \frac{dz}{dx_n} = R$$

form the system of ordinary differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} \ldots = \frac{dx_n}{X_n} = \frac{dz}{R},$$

and deduce their general integrals

$$u_1 = a_1, \ u_2 = a_2, \ldots u_n = a_n,$$

then

$$F(u_1, u_2, \ldots u_n) = 0$$

will be the general integral sought.

[The *general observations* were intended to follow Chap. XIV. Art. 6.]

General observations.

4. The relation which exists between a proposed linear partial differential equation and its auxiliary system of ordinary differential equations should be carefully studied. While it is proper to say as above that the general integral of the one requires the knowledge of all the integrals of the other, it is also proper to describe that general integral simply as the most general form under which an integral of the auxiliary system can appear. If

$$u_1 = a_1, \ u_2 = a_2, \ldots u_n = a_n$$

are integrals of that system, then

$$F(u_1, u_2, \ldots u_n) = A$$

is the one general form of an integral of that system, and due regard being had to the arbitrariness of F, this is equivalent to

$$F(u_1, u_2, \dots u_n) = 0.$$

5. The form which the auxiliary system assumes when the given partial differential equation is deficient in any of its terms should be noticed.

If $X_1 = 0$, the auxiliary equation

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2}$$

becomes, on clearing of fractions,

$$dx_1 = 0.$$

And thus, if $X_1, X_2, \dots X_r$ vanish, the given equation being

$$X_1 \frac{dz}{dx_1} \dots + X_{r+1} \frac{dz}{dx_{r+1}} \dots + X_n \frac{dz}{dx_n} = X,$$

the auxiliary system will be

$$dx_1 = 0, \quad dx_2 = 0, \dots dx_r = 0,$$

$$\frac{dx_{r+1}}{X_{r+1}} = \frac{dx_{r+2}}{X_{r+2}} \dots = \frac{dx_n}{X_n} = \frac{dz}{X},$$

and the integrals of this system being of the form

$$x_1 = a_1, \quad x_2 = a_2, \dots x_r = a_r,$$

$$u_{r+1} = a_{r+1}, \dots\dots u_n = a_n,$$

the general solution of the given equation will be

$$F(x_1, \dots\dots x_r, u_{r+1}, \dots\dots u_n) = 0.$$

This conclusion would follow also from the principle laid down in Chap. XIV. Art. 2.

Linear partial differential equations in which the absolute term is wanting, and which are therefore of the form

$$X_1 \frac{dz}{dx_1} + X_2 \frac{dz}{dx_2} \dots + X_n \frac{dz}{dx_n} = 0,$$

may be termed homogeneous. As in this case one of the auxiliary equations is

$$dz = 0,$$

the general integral will be

$$F(u_1, u_2, \ldots\ldots u_{n-1}, z) = 0,$$

$u_1, u_2, \ldots\ldots u_{n-1}$ being found by the integration of the remaining auxiliary equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} \cdots = \frac{dx_n}{X_n}.$$

When $X_1, X_2, \ldots\ldots X_n$ do not contain z, the solution is best exhibited in the form

$$z = \phi(u_1, u_2, \ldots\ldots u_{n-1}).$$

6. Every linear partial differential equation can be converted into a homogeneous one containing one additional variable. For it is shewn in Art. 3, that if $u = 0$ be any integral of

$$X_1 \frac{dz}{dx_1} + X_2 \frac{dz}{dx_2} \cdots + X_n \frac{dz}{dx_n} = X,$$

then is

$$X_1 \frac{du}{dx_1} + X_2 \frac{du}{dx_2} \cdots + X_n \frac{du}{dx_n} + X \frac{du}{dz} = 0,$$

a homogeneous equation with a new variable.

From the general integral of this equation, that of the former one may be deduced by making $u = 0$.

7. The solution of partial differential equations is sometimes facilitated by introducing a new system of independent variables. The actual transformation is greatly facilitated by the following symbolical theorem.

THEOREM. If the partial differential equation

$$X_1 \frac{dz}{dx_1} + X_2 \frac{dz}{dx_2} \cdots + X_n \frac{dz}{dx_n} = X$$

be expressed symbolically in the form

$$\Delta z = X,$$

in which

$$\Delta = X_1 \frac{d}{dx_1} + X_2 \frac{d}{dx_2} \dots + X_n \frac{d}{dx_n},$$

then, if $y_1, y_2, \dots\dots y_n$ be a new system of independent variables given in expression as functions of the old ones, the transformed equation will be

$$(\Delta y_1)\frac{dz}{dy_1} + (\Delta y_2)\frac{dz}{dy_2} \dots + (\Delta y_n)\frac{dz}{dy_n} = X.$$

For, regarding z as a function of $y_1, y_2, \dots\dots y_n$, we have

$$\frac{dz}{dx_1} = \frac{dz}{dy_1}\frac{dy_1}{dx_1} + \frac{dz}{dy_2}\frac{dy_2}{dx_1} \dots + \frac{dz}{dy_n}\frac{dy_n}{dx_1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{dz}{dx_n} = \frac{dz}{dy_1}\frac{dy_1}{dx_n} + \frac{dz}{dy_2}\frac{dy_2}{dx_n} \dots + \frac{dz}{dy_n}\frac{dy_n}{dx_n};$$

whence, substituting in the given equation we find, as the total coefficient of $\frac{dz}{dy_1}$, the expression

$$X_1\frac{dy_1}{dx_1} + X_2\frac{dy_1}{dx_2} \dots + X_n\frac{dy_1}{dx_n},$$

or symbolically, Δy_1; and so on for the other coefficients. The result then is

$$(\Delta y_1)\frac{dz}{dy_1} + (\Delta y_2)\frac{dz}{dy_2} \dots + (\Delta y_n)\frac{dz}{dy_n} = X.$$

It remains only after calculation of $\Delta y_1, \Delta y_2, \dots\dots \Delta y_n$, as functions of $x_1, x_2, \dots\dots x_n$, to express these functions and X in terms of $y_1, y_2, \dots\dots y_n$.

[It appears from the manuscript that an example was to have been supplied here.]

[The next Article may be considered supplementary to Chap. XIV. Art. 10.]

Singular Solutions of partial Differential Equations.

8. Legendre's theory developed in Chap. XXIII. for ordinary, may be applied also without essential change to partial, differential equations. Regarding the independent variable z as receiving an infinitesimal change δz through infinitesimal change, not in the values of the independent variables

$$x_1, \; x_2, \ldots\ldots x_n,$$

but in the values of the arbitrary constants of the complete or in the forms of the arbitrary functions of the general integral, and performing upon the given equation the operation denoted by δ, we shall obtain a *linear* partial differential equation for determining the general value of δz corresponding to any particular given value of z. If that linear equation be of a lower order than the differential equation given, then the equation expressing the value of $z + \delta z$ will be a limiting form of a solution less complete or less general than the complete or general solution of the differential equation given, and the given solution, formed by making the infinitesimal constants in the limiting form actually 0, will be singular.

Conversely, to deduce singular solutions without the knowledge of the complete or the general integral, we ought to construct the equations of condition for the reduction of the equation determining δz to a lower order than the equation given, and the most general solution of the differential equations of condition so formed, will be the most general expression for the singular solutions of the differential equation given.

Ex. $$(px - qy)^2 q + 4mx^3 (z - xp) = 0,$$

in which

$$p = \frac{dz}{dx}, \quad q = \frac{dz}{dy}.$$

Representing the first member of the equation by F, we have, on operating by δ,

$$\frac{dF}{dp}\frac{d\delta z}{dx}+\frac{dF}{dq}\frac{d\delta z}{dy}+\frac{dF}{dz}\delta z=0,$$

and the conditions

$$\frac{dF}{dp}=0, \quad \frac{dF}{dq}=0,$$

necessary to reduce the equation for δz to a lower order give

$$(px-qy)\,q-2mx^3=0,$$

$$(px-qy)\,(px-3qy)=0.$$

From these we find

$$p=3m^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}}, \quad q=m^{\frac{1}{2}}x^{\frac{3}{2}}y^{-\frac{1}{2}},$$

definite and simultaneous values of p and q, which being substituted in the given equation lead to

$$z=2m^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{1}{2}},$$

and this, as it gives the same values of p and q as those obtained before, will necessarily satisfy the given equation. It is therefore a solution, and from the nature of the analysis, a singular one.

Legendre shews that this singular solution is also deducible from the general integral of the given partial differential equation. That integral is the result of the elimination of a from the two equations

$$\{\phi(a)\}^2-2ax\phi(a)+az-mxy=0,$$

$$\{\phi(a)-ax\}\,\phi'(a)-2x\phi(a)+z=0.$$

To deduce the singular solution he supposes $\phi(a)$ to be *not* simply a function of a, but a function of a and of one or both of the independent variables. He expresses the variation of $\phi(a)$ derived from this new source by δ, and operating on the first equation with δ, finds

$$\{2\phi(a) - 2ax\}\, \delta\phi(a) = 0;$$

therefore $$\phi(a) = ax.$$

Substituting this in the equations of the general integral, and eliminating a, we find

$$z = 2m^{\frac{1}{3}}x^{\frac{2}{3}}y^{\frac{1}{3}}$$

as before.

Legendre states his theory of the derivation of the singular solutions of partial differential equations from the equations themselves with great brevity, but still as a *general* theory. And there is nothing in the statement that carries with it any apparent restriction upon either the order or the degree of the equations given. Until however we are in possession of a perfect theory of the *genesis* of partial differential equations we shall not be entitled to say that Legendre's theory of their singular solutions is a perfect one; for until then we cannot even define, in a perfectly general way, the nature of the operation denoted by δ.

[The next three Chapters all relate to the subject of partial differential equations of the first order. The manuscripts do not appear to have received their final revision from Professor Boole. It is certain that he intended the contents of Chapter XXV. to form a part of the new edition; and it is highly probable, although not certain, that the contents of Chapter XXVI. and Chapter XXVII. were also to be included.

The three Chapters are mainly derived from two memoirs by Professor Boole, published in the *Philosophical Transactions.*

The first memoir is entitled *On Simultaneous Differential Equations of the First Order in which the Number of the Variables exceeds by more than one the Number of the Equations:* it occupies pages 437...454 of the *Philosophical Transactions* for 1862.

The second memoir is entitled *On the Differential Equations of Dynamics. A sequel to a Paper on Simultaneous Differential Equations:* it occupies pages 485...501 of the *Philosophical Transactions* for 1863.

The first memoir was finished before Professor Boole had seen Jacobi's researches, which are cited at the beginning of Chapter XXVI; these researches indeed could only just have been published. In his second memoir Professor Boole describes Jacobi's methods, refers to his own already published, and points out the nature of the connexion between them.]

CHAPTER XXV.

ON SYSTEMS OF SIMULTANEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER, AND ON ASSOCIATED SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS.

1. The term *simultaneous* is here applied to a system of partial differential equations, to signify that in that system there is but one dependent variable, the general expression of which, as a function of the independent variables satisfying all the equations at once, is the object of search. All linear partial differential equations of the first order being reducible to the homogeneous form, we shall presuppose this reduction here. Under this form indeed the problem actually presents itself in Geometry, in the theory of partial differential equations of the second order, and in Theoretical Dynamics.

We are sometimes led, in connexion with the same class of inquiries, to systems of ordinary differential equations marked by the peculiarity that the number of the variables exceeds by more than one the number of the equations. Such systems are intimately connected with the former—stand to them indeed in a similar relation to that which the Lagrangean auxiliary system bears to the *single* partial differential equation from which it arises. The theory which explains this connexion, and grounds upon it the method of solution of both systems will form the subject of the present Chapter.

Connexion of the Systems.

2. Prop. I. The solution of a system of simultaneous linear partial differential equations of the first order may be

made to depend upon that of a system of ordinary differential equations of the first order in which the number of the variables exceeds by more than one the number of the equations.

The system of partial differential equations being reduced to the homogeneous form, Chap. XXIV. Art. 6, let n be the number of the equations, x_1, x_2, ... x_{n+r} the independent variables, and P the dependent variable.

Then from the n given equations determining

$$\frac{dP}{dx_1}, \frac{dP}{dx_2}, \ldots\ldots \frac{dP}{dx_n},$$

we obtain an equivalent system of equations which, by transposition of its terms to one side, assumes the reduced form

$$\left.\begin{array}{l} \dfrac{dP}{dx_1} + A_{11}\dfrac{dP}{dx_{n+1}} + A_{12}\dfrac{dP}{dx_{n+2}} \ldots\ldots + A_{1r}\dfrac{dP}{dx_{n+r}} = 0 \\ \dfrac{dP}{dx_2} + A_{21}\dfrac{dP}{dx_{n+1}} + A_{22}\dfrac{dP}{dx_{n+2}} \ldots\ldots + A_{2r}\dfrac{dP}{dx_{n+r}} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dP}{dx_n} + A_{n1}\dfrac{dP}{dx_{n+1}} + A_{n2}\dfrac{dP}{dx_{n+2}} \ldots\ldots + A_{nr}\dfrac{dP}{dx_{n+r}} = 0 \end{array}\right\} \ldots\ldots(1),$$

Multiplying these equations by the arbitrary constants

$$\lambda_1, \lambda_2, \ldots\ldots \lambda_n$$

respectively, and adding the results, we have

$$\begin{aligned} &\lambda_1\frac{dP}{dx_1} + \lambda_2\frac{dP}{dx_2} \ldots\ldots + \lambda_n\frac{dP}{dx_n} \\ &\quad + (\lambda_1A_{11} + \lambda_2A_{21} \ldots\ldots + \lambda_nA_{n1})\frac{dP}{dx_{n+1}} \\ &\quad + \ldots\ldots \\ &\quad + (\lambda_1A_{1r} + \lambda_2A_{2r} \ldots\ldots + \lambda_nA_{nr})\frac{dP}{dx_{n+r}} = 0 \ldots\ldots (2), \end{aligned}$$

a single partial differential equation which, on account of the arbitrariness of $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$, is equivalent to the system from which it was formed.

Of this equation the Lagrangean auxiliary system will be

$$\frac{dx_1}{\lambda_1} = \frac{dx_2}{\lambda_2} \ldots\ldots = \frac{dx_n}{\lambda_n}$$

$$= \frac{dx_{n+1}}{\lambda_1 A_{11} + \lambda_2 A_{21} \ldots\ldots + \lambda_n A_{n1}}$$

$$= \ldots\ldots$$

$$= \frac{dx_{n+r}}{\lambda_1 A_{1r} + \lambda_2 A_{2r} \ldots\ldots + \lambda_n A_{nr}} \ldots\ldots\ldots\ldots\ldots (3),$$

whence, eliminating $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$, we have the system of ordinary differential equations

$$\left.\begin{array}{l} dx_{n+1} - A_{11}dx_1 - A_{21}dx_2 \ldots . - A_{n1}dx_n = 0 \\ dx_{n+2} - A_{12}dx_1 - A_{22}dx_2 \ldots . - A_{n2}dx_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dx_{n+r} - A_{1r}dx_1 - A_{2r}dx_2 \ldots . - A_{nr}dx_n = 0 \end{array}\right\} \ldots\ldots\ldots (4),$$

These equations being included in the previous system (3), any integrals

$$u = a, \quad v = b, \quad w = c, \quad \&c.$$

of them will be integrals of it. Therefore $u, v, w, \ldots$ will be values of P satisfying the partial differential equation (2). For they will be the only values which can satisfy it independently of $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$. Hence they will satisfy the equivalent system (1), and the general integral of that system will be

$$F(u, v, w, \ldots) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

the form of F being arbitrary.

Thus the relation of the system (4) to the system (1) is the same as the relation of the auxiliary system of a single linear partial differential equation to that equation. And the ground of this relation is seen to be the same in both cases. The one form necessitates the other.

3. Instead of employing the above mode of deducing the auxiliary system, we might employ the following which is practically more convenient.

Since any value P which satisfies the partial differential equations determines $P=c$ as an integral of the ordinary system, the latter must be consistent with $dP=0$ in its developed form

$$\frac{dP}{dx_1}dx_1+\frac{dP}{dx_2}dx_2 \ldots\ldots + \frac{dP}{dx_{n+r}}dx_{n+r}=0.$$

Eliminating $$\frac{dP}{dx_1},\ \frac{dP}{dx_2},\ldots\ldots\frac{dP}{dx_n}$$

by means of the n given equations (1), we have

$$\frac{dP}{dx_{n+1}}(dx_{n+1}-A_{11}dx_1-A_{21}dx_2\ldots\ldots-A_{n1}dx_n)$$

$$+\frac{dP}{dx_{n+2}}(dx_{n+2}-A_{12}dx_1-A_{22}dx_2\ldots\ldots-A_{n2}dx_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{dP}{dx_{n+r}}(dx_{n+r}-A_{1r}dx_1-A_{2r}dx_2\ldots\ldots-A_{nr}dx_n)=0.$$

Whence, equating to 0 the respective coefficients of

$$\frac{dP}{dx_{n+1}},\ \frac{dP}{dx_{n+2}},\ldots\ldots\frac{dP}{dx_{n+r}},$$

we have the system (4).

In the same way we can pass from the system of ordinary to that of partial differential equations. *From the equation* $dP=0$, *in its developed form, we must eliminate a number* *of differentials dx_1, dx_2, ... equal to that of the given equations, and then equate to 0 the coefficients of the remaining differentials.*

4. Lastly, the formal connexion of the two systems should be noticed. The partial differential equations being given in the reduced form (1), the ordinary system may be constructed as follows: For any differential coefficient, as $\frac{dP}{dx_{n+1}}$, in any column after the first, write the corresponding differential dx_{n+1}, subtract from this the sum of dx_1, dx_2, ... dx_n, multiplied respectively by the descending coefficients of that column, and equate the result to 0. The system of equations thus successively formed will be the auxiliary system sought.

The transition from the ordinary to the partial system may be effected by the same rule, substituting only differentials for differential coefficients.

[It appears from the manuscript that an example was to have been supplied here.]

Up to this point the theory of systems of partial differential equations is in analogy with that of single equations. But here a difference arises. We do not know beforehand what number of integrals a system of ordinary differential equations, in which the number of variables exceeds by more than one the number of the equations, admits.

The theory which removes this difficulty will be developed in the following sections. It will be shewn that a system of linear partial differential equations which admits of solution by the assigning to the dependent variable a value which satisfies all the equations in common, must either itself satisfy a certain condition, or be capable of being developed into a new but equivalent system which will satisfy that condition. It will be shewn that when that condition is satisfied, the auxiliary system of ordinary, is capable of expression as a system of *exact* differential equations determining the integrals sought.

It will be found convenient to express by a single symbol the aggregate of the operations to which the dependent variable is subject in the expression of a partial differential equation. Thus the equation

$$\frac{dz}{dt} + x\frac{dz}{dx} + y\frac{dz}{dy} = 0$$

may be expressed in the form

$$\Delta z = 0$$

if we assume

$$\Delta = \frac{d}{dt} + x\frac{d}{dx} + y\frac{d}{dy}.$$

Under this convention the following proposition is to be understood.

5. Prop. II. If $\Delta P = 0$, $\Delta' P = 0$ represent any two homogeneous linear partial differential equations of the first order, then will

$$(\Delta\Delta' - \Delta'\Delta) P = 0$$

also be a homogeneous linear partial differential equation of the first order, and it will be satisfied by all the common integrals of the equations from which it is derived.

First, the equation will be linear. For, let x, y represent any two variables whatever, or the same variable repeated, out of the set $x_1 \ldots. x_n$, and let A, B represent any functions of the variables $x_1 \ldots. x_n$. Then Δ may be represented by a series of terms of the form $A\dfrac{d}{dx}$, and Δ' by a series of terms of the form $B\dfrac{d}{dy}$. Hence $(\Delta\Delta' - \Delta'\Delta) P$ can be expressed by a series of terms of the form

$$A\frac{d}{dx}\left(B\frac{dP}{dy}\right) - B\frac{d}{dy}\left(A\frac{dP}{dx}\right),$$

which, on effecting the differentiations, becomes

$$A\frac{dB}{dx}\frac{dP}{dy} - B\frac{dA}{dy}\frac{dP}{dx},$$

the terms containing the second differential coefficients of P mutually destroying each other. Hence the equation

$$(\Delta\Delta' - \Delta'\Delta)P = 0$$

will be a homogeneous linear partial differential equation of the first order.

The constitution of the coefficients of this equation is easily determined. For suppose the given equations to be

$$A_1\frac{dP}{dx_1} + A_2\frac{dP}{dx_2} \ldots\ldots + A_n\frac{dP}{dx_n} = 0,$$

$$B_1\frac{dP}{dx_1} + B_2\frac{dP}{dx_2} \ldots\ldots + B_n\frac{dP}{dx_n} = 0,$$

so that

$$\Delta = A_1\frac{d}{dx_1} \ldots\ldots + A_n\frac{d}{dx_n}, \quad \Delta' = B_1\frac{d}{dx_1} \ldots\ldots + B_n\frac{d}{dx_n},$$

then the equation

$$(\Delta\Delta' - \Delta'\Delta)\,P = 0$$

may be written in the form

$$\Delta\left(B_1\frac{dP}{dx_1} + B_2\frac{dP}{dx_2} \ldots\ldots + B_n\frac{dP}{dx_n}\right)$$

$$- \Delta'\left(A_1\frac{dP}{dx_1} + A_2\frac{dP}{dx_2} \ldots\ldots + A_n\frac{dP}{dx_n}\right),$$

and, since terms involving second differential coefficients of P will disappear, this becomes

$$(\Delta B_1 - \Delta' A_1)\frac{dP}{dx_1} + (\Delta B_2 - \Delta' A_2)\frac{dP}{dx_2} \ldots\ldots$$

$$+ (\Delta B_n - \Delta' A_n)\frac{dP}{dx_n} = 0.$$

We see from this that the *form of the result is the same as if the* Δ *or* Δ' *from either equation operated only on the coefficients in the other equation.*

Secondly, the above equation will be satisfied by all the common integrals of the equations from which it is derived.

For, let $\phi = c$ be a common integral of

$$\Delta P = 0 \text{ and } \Delta' P = 0,$$

then

$$\Delta\phi = 0, \qquad \Delta'\phi = 0.$$

Performing on these the respective operations Δ' and Δ, operations which involve only differentiation together with algebraic processes, we have

$$\Delta'\Delta\phi = 0, \quad \Delta\Delta'\phi = 0,$$

whence, by subtraction,

$$\Delta\Delta'\phi - \Delta'\Delta\phi = 0,$$

or

$$(\Delta\Delta' - \Delta'\Delta)\,\phi = 0,$$

from which it appears that ϕ is also an integral of the equation

$$(\Delta\Delta' - \Delta'\Delta)\,P = 0,$$

as was to be shewn.

6. PROP. III. If by the above processes of reduction and derivation we convert a system of partial differential equations into a new system, such that if expressed in the form

$$\Delta_1 P = 0, \quad \Delta_2 P = 0, \ldots\ldots \Delta_m P = 0,$$

the condition

$$(\Delta_i\Delta_j - \Delta_j\Delta_i)\,P = 0$$

shall for each pair of equations be identically satisfied, then the system of ordinary differential equations corresponding to this new system will admit of reduction to the form of *exact* differential equations, the integration of which will enable us to construct the general value of P satisfying the system given.

1st. Suppose the given system of n equations reduced to the form (1), marked by the peculiarity that n of the differential coefficients appear only in successive equations and with the coefficient unity. Then taking any two of those equations (we select the first two), we have

$$\Delta_1 = \frac{d}{dx_1} + A_{11}\frac{d}{dx_{n+1}} \dots\dots + A_{1r}\frac{d}{dx_{n+r}},$$

$$\Delta_2 = \frac{d}{dx_2} + A_{21}\frac{d}{dx_{n+1}} \dots\dots + A_{2r}\frac{d}{dx_{n+r}},$$

from the forms of which we see that the derived equation

$$(\Delta_1\Delta_2 - \Delta_2\Delta_1)P = 0$$

cannot contain either

$$\frac{dP}{dx_1} \text{ or } \frac{dP}{dx_2}.$$

It can only, as appears from Art. 5, contain the differential coefficients

$$\frac{dP}{dx_{n+1}}, \dots\dots \frac{dP}{dx_{n+r}},$$

and must be of the form

$$B_1\frac{dP}{dx_{n+1}} + B_2\frac{dP}{dx_{n+2}} \dots\dots + B_r\frac{dP}{dx_{n+r}} = 0.$$

It cannot therefore be an *algebraic* consequence of any of the equations of the system (1) from which it was derived. It is, unless by the vanishing of $B_1, \dots . B_r$ it present itself as an identity, a *new* equation algebraically independent. Combining this with the former ones, we have a system of $n+1$ equations admitting of the same reduction as to form followed by the same subsequent process of derivation. And the result of each of these completed steps is to convert the system into one containing one equation more than before; but containing in each of its equations one term fewer than before. The process must then end either in the genesis of a system of partial differential equations such that the further application of the process of derivation of Prop. II. shall only lead to identities, or in the emerging of the system

$$\frac{dP}{dx_1}=0,\ \frac{dP}{dx_2}=0, \ldots\ldots \frac{dP}{dx_{n+r}}=0.$$

The latter supposition would imply that P is a constant. The consequences of the former we proceed to examine.

The final system of linear partial differential equations will be of the same type (1) as the original system, but will differ from that system in that n will be increased, and r diminished by the same amount. We shall therefore simply state the form (1), only under the condition

$$(\Delta_i\Delta_j - \Delta_j\Delta_i)\, P = 0,$$

and with the altered values of n and r.

First, then, the common integrals of the new system will be the same as those of the original system. This is evident from Prop. II.

Secondly. If we write

$$m = n + r,$$

the first equation of the system (1) will be

$$\frac{dP}{dx_1} + A_{11}\frac{dP}{dx_{n+1}} + A_{12}\frac{dP}{dx_{n+2}} \ldots\ldots + A_{1r}\frac{dP}{dx_m} = 0,$$

and the auxiliary Lagrangean system of this will have $m-1$ independent integrals

$$u_1 = c_1,\quad u_2 = c_2, \ldots\ldots u_{m-1} = c_{m-1},$$

among which the $n-1$ known integrals (Chap. XXIV. Art. 5)

$$x_2 = c_2,\quad x_3 = c_3,\quad x_n = c_n$$

are included. And the general value of P satisfying the above first equation will be

$$P = F(u_1, u_2 \dots u_{m-1}).$$

The assumption $P = x_1$ would not satisfy the said equation, for it would lead, on substitution, to $1 = 0$. Hence we infer that while the functions $u_1, u_2, \dots\dots u_{m-1}$ are independent with respect to each other, they are also independent with respect to x_1, so that the m functions $u_1, u_2, \dots\dots u_{m-1}, x_1$, are mutually independent in the sense explained in Chap. XXIV.

Let us now transform the equations of the system (1) after the first by introducing $u_1, u_2, \dots\dots u_{m-1}, x_1$ as independent variables. Those equations being

$$\Delta_2 P = 0, \dots\dots \Delta_n P = 0,$$

the result of the transformation will be (Chap. XXIV. Art. 7)

$$(\Delta_2 u_1)\frac{dP}{du_1} + (\Delta_2 u_2)\frac{dP}{du_2} \dots\dots + (\Delta_2 u_{m-1})\frac{dP}{du_{m-1}} + (\Delta_2 x_1)\frac{dP}{dx_1} = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(\Delta_n u_1)\frac{dP}{du_1} + (\Delta_n u_2)\frac{dP}{du_2} \dots\dots + (\Delta_n u_{m-1})\frac{dP}{du_{m-1}} + (\Delta_n x_1)\frac{dP}{dx_1} = 0.$$

But $P = x_1$ being an integral of each of the equations of the system (1) except the first, as appears from their forms, we have

$$\Delta_2 x_1 = 0, \dots\dots \Delta_n x_1 = 0,$$

thus the last terms in the transformed system vanish. Further, the coefficients of the remaining terms reduce to functions of $u_1, u_2, \dots\dots u_{m-1}$ merely. For, considering the coefficient $\Delta_2 u_1$, we have

$$(\Delta_1 \Delta_2 - \Delta_2 \Delta_1) u_1 = 0,$$

which, since $\Delta_1 u_1 = 0$ reduces to

$$\Delta_1 \Delta_2 u_1 = 0.$$

Hence $\Delta_2 u_1$ must be a solution of $\Delta_1 P = 0$, and therefore a function of $u_1, u_2, \ldots\ldots u_{m-1}$. And so for the others. It results therefore that the transformed system is

$$(\Delta_2 u_1)\frac{dP}{du_1} + (\Delta_2 u_2)\frac{dP}{du_2} \ldots\ldots + (\Delta_2 u_{m-1})\frac{dP}{du_{m-1}} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(\Delta_n u_1)\frac{dP}{du_1} + (\Delta_n u_2)\frac{dP}{du_2} \ldots\ldots + (\Delta_n u_{m-1})\frac{dP}{du_{m-1}} = 0,$$

$u_1, u_2, \ldots\ldots u_{m-1}$ being the *actual* independent variables of the system.

But the transformation having involved no loss of generality, for a new system of m independent variables was simply substituted for an old one, the condition

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0,$$

satisfied before, will continue to be satisfied in the new system represented symbolically in the form

$$\Delta_2 P = 0, \quad \Delta_3 P = 0, \ldots\ldots \Delta_n P = 0.$$

Any common integrals of this system will also be common integrals of the previous system. For as functions of

$$u_1, u_2, \ldots\ldots u_{m-1}$$

they will satisfy the first equation of that system, and they will satisfy the other equations, because the present system is but a transformation of those. The converse is equally manifest.

Thus a system of n partial differential equations containing m independent variables and satisfying a certain condition, has in virtue of that condition been converted into a system of $n-1$ equations between $m-1$ independent variables, and satisfying the same condition. This then is convertible into a similarly constituted system of $n-2$ equations containing $m-2$ independent variables, and so on till we arrive at a final single partial differential equation containing $m-n+1$ independent variables. This equation has $m-n$, that is, r integrals, and these are the common integrals of the system (1).

But the system of *ordinary* differential equations corresponding to (1) is in number r, and is satisfied by all the common integrals of that system. Hence these differential equations must admit of reduction to the exact form.

7. We may deduce from the above investigation the following Rule.

To integrate a system of simultaneous linear partial differential equations of the first order.

RULE. Reduce the equations to the homogeneous form (1), express the result symbolically by

$$\Delta_1 P=0, \quad \Delta_2 P=0, \ldots\ldots \Delta_n P=0,$$

and examine whether the condition

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)\,P=0$$

is identically satisfied for every pair of equations of the system. If it be so, the equations of the auxiliary system, Prop. I., will be reducible to the exact form, and their integrals being

$$u=a, \quad v=b, \quad w=c, \ldots\ldots$$

the complete value of P will be $F(u, v, w, \ldots)$, the form of F being arbitrary.

If the condition be not identically satisfied, its application will give rise to one or more new partial differential equations. Combine any one of these with the previous reduced system, and again reduce in the same way. With the new reduced system proceed as before, and continue this method of reduction and derivation until either a system of partial differential equations arises between every two of which the above condition is identically satisfied, or, which is the only possible alternative, the system

$$\frac{dP}{dx_1}=0, \quad \frac{dP}{dx_2}=0, \ldots$$

appears. In the former case the system of ordinary equations corresponding to the final system of partial differential equations will admit of reduction to the exact form, and the general value of P will emerge from their integrals as above. In the latter case the given system can only be satisfied by supposing P a constant.

Ultimately then the determination of P depends on the solution of a system of ordinary differential equations reducible to the exact form. This does not mean that each equation of the system is reducible to the exact form, but that the equations may be combined together so as to form an equal number of equivalent equations of the exact form. Generally when we know this combination to be possible it is easy to effect it, and best to endeavour to do so. We might however employ the method of the variation of parameters as follows. Supposing p the number of differential equations make all but $p+1$ of the variables constant, integrate the reduced system, and then seek to satisfy the unreduced system by the same series of integrals with the arbitrary constants as new variables. The successive integrations and transformations of this method would amount to the same thing as those upon which the second part of the demonstration of Prop. III. rests*.

Lastly, given a system of ordinary differential equations containing a superfluous number of variables without knowing how many integrals they admit, we must, supposing $P=c$ to be *any* integral, construct the corresponding system

* It was thus indeed that the author was first led to that theory.

of homogeneous partial differential equations satisfied by P, and apply to them the foregoing Rule.

8. Ex. Required the integrals of the simultaneous partial differential equations

$$\frac{dP}{dx} + (t + xy + xz)\frac{dP}{dz} + (y + z - 3x)\frac{dP}{dt} = 0,$$

$$\frac{dP}{dy} + (xzt + y - xy)\frac{dP}{dz} + (zt - y)\frac{dP}{dt} = 0.$$

Representing these in the form $\Delta_1 P = 0$, $\Delta_2 P = 0$, it will be found that the equation

$$(\Delta_1\Delta_2 - \Delta_2\Delta_1)\, P = 0$$

becomes, after rejecting an algebraic factor,

$$x\frac{dP}{dz} + \frac{dP}{dt} = 0,$$

and the three equations prepared in the manner explained in the Rule will be found to be

$$\frac{dP}{dx} + (3x^2 + t)\frac{dP}{dz} = 0,$$

$$\frac{dP}{dy} + y\frac{dP}{dz} = 0,$$

$$\frac{dP}{dt} + x\frac{dP}{dz} = 0.$$

No other equations are derivable from these. We conclude that there is but one final integral.

To obtain it, eliminate

$$\frac{dP}{dx}, \quad \frac{dP}{dy}, \quad \frac{dP}{dt}$$

from the above system combined with

$$\frac{dP}{dx}dx + \frac{dP}{dy}dy + \frac{dP}{dz}dz + \frac{dP}{dt}dt = 0,$$

and equate to 0 the coefficient of $\frac{dP}{dz}$ in the result. We find

$$dz - (t + 3x^2)\,dx - ydy - xdt = 0,$$

the integral of which is

$$z - xt - x^3 - \frac{y^2}{2} = c.$$

An arbitrary function of the first member of this equation is the general value of P.

[It appears from the manuscript that another example was to have been added here.]

CHAPTER XXVI.

HOMOGENEOUS SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

1. THE theory of homogeneous systems of linear partial differential equations in which when expressed in the symbolic form

$$\Delta_1 P = 0, \quad \Delta_2 P = 0, \ldots\ldots \Delta_m P = 0 \ldots\ldots\ldots(1),$$

the condition

$$(\Delta_i \Delta_j - \Delta_j \Delta_i)\, P = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is for all combinations represented by i and j satisfied in virtue of the constitution of the symbols Δ_i, Δ_j, forms the subject of important researches by Jacobi (*Nova Methodus*... Crelle's Journal, Vol. LX. p. 1). The following are the most important of his results.

1st. An integral of any one equation of the system being found, other integrals of the same system may be obtained without integration, by a process of derivation founded upon the condition (2).

Let ϕ be an integral of the first equation of the system. Then is the equation

$$\Delta_1 \phi = 0$$

identically satisfied.

Also the condition (2) being satisfied in virtue of the constitution of the symbols, we have

$$(\Delta_i \Delta_j - \Delta_j \Delta_i)\, \phi = 0;$$

and in particular, making $i=1$, and separating the terms,

$$\Delta_1\Delta_j\phi - \Delta_j\Delta_1\phi = 0,$$

which reduces by a prior equation to

$$\Delta_1\Delta_j\phi = 0.$$

It appears from this that $\Delta_j\phi$, if it do not reduce to a constant, is an integral of the first equation $\Delta_1\phi = 0$, and, if it prove to be not a mere function of ϕ, a new integral.

This process may be repeated upon the new integral with a similar alternation of results. It will be evident from this that if we confine our attention to the two equations

$$\Delta_1 P = 0, \quad \Delta_2 P = 0,$$

and suppose, as before, ϕ to be an integral of the first, then will

$$\Delta_2\phi, \quad \Delta_2(\Delta_2\phi), \quad \Delta_2\{\Delta_2(\Delta_2\phi)\}, \ldots$$

or, as these may be expressed,

$$\Delta_2\phi, \quad \Delta_2^2\phi, \quad \Delta_2^3\phi, \ldots$$

be also integrals of the first equation; and this process of derivation may be continued until we arrive at an integral $\Delta_2^\mu\phi$ which is not independent, but is expressible as a function of prior integrals

$$\Delta_2\phi, \quad \Delta_2^2\phi, \ldots\ldots \Delta_2^{\mu-1}\phi,$$

and, sooner or later, such a result must present itself, since the number of independent integrals is finite.

It is further seen that the most general symbolic form of an integral derivable from the root integral ϕ is

$$\Delta_2^\alpha\Delta_3^\beta \ldots\ldots \Delta_m^\mu\phi,$$

$\alpha, \beta, \ldots\ldots \mu$, being positive integers.

The above remarkable theorem was in some degree anticipated by the researches of Poisson.

2ndly. Jacobi shews how by the aid of such derived integrals of the first equation of the system a *common* integral of the first and second equation may be found, and how from this integral and its derived series a common integral of the first three equations of the system may be found, and so on, until a common integral of the entire system has been as it were built up out of previous integrals of less general application.

Let $\phi, \phi', \phi'', \ldots\ldots \phi^{(\mu-1)}$ represent a series of independent integrals of the equation $\Delta_1 P = 0$, of which ϕ is the root integral, and the rest are derived from it by successive applications of the operation denoted by Δ_2, so that

$$\phi' = \Delta_2\phi, \ldots\ldots \phi^{(\mu-1)} = \Delta_2^{\mu-1}\phi;$$

also let $\Delta_2^{\mu}\phi$ be not a new integral but a function of

$$\phi, \phi', \ldots\ldots \phi^{(\mu-1)}.$$

Now $\phi, \phi', \ldots\ldots \phi^{(\mu-1)}$ being particular integrals of $\Delta_1 P = 0$, the function $F(\phi, \phi', \ldots\ldots \phi^{(\mu-1)})$ will also be an integral of the same equation *irrespectively of its form.* Let us inquire whether the form of the function can be so determined as to render it also an integral of the second equation $\Delta_2 P = 0$.

We have then to satisfy the equation

$$\Delta_2 F(\phi, \phi', \ldots\ldots \phi^{(\mu-1)}) = 0.$$

By the principles of the Differential Calculus this equation assumes the form

$$\Delta_2\phi \frac{dF}{d\phi} + \Delta_2\phi' \frac{dF}{d\phi'} + \ldots\ldots + \Delta_2\phi^{(\mu-1)} \frac{dF}{d\phi^{(\mu-1)}} = 0.$$

But $\qquad \Delta_2\phi = \phi', \; \Delta_2\phi' = \phi'', \ldots\ldots \Delta_2\phi^{(\mu-2)} = \phi^{(\mu-1)};$

lastly, $\Delta_2\phi^{(\mu-1)}$ may by hypothesis be expressed in the form $f(\phi, \phi', \ldots\ldots \phi^{(\mu-1)})$. Thus the equation to be satisfied is

$$\phi'\frac{dF}{d\phi}+\phi''\frac{dF}{d\phi'}+\ldots\ldots+\phi^{(\mu-1)}\frac{dF}{d\phi^{(\mu-2)}}$$

$$+f(\phi, \phi', \ldots\ldots \phi^{(\mu-1)})\frac{dF}{d\phi^{(\mu-1)}}=0,$$

a linear partial differential equation of which the auxiliary system is

$$\frac{d\phi}{\phi'}=\frac{d\phi'}{\phi''}=\ldots\ldots=\frac{d\phi^{(\mu-2)}}{\phi^{(\mu-1)}}=\frac{d\phi^{(\mu-1)}}{f(\phi, \phi', \ldots\ldots \phi^{(\mu-1)})} \ldots. (3).$$

Now the integration of this system may be made to depend upon that of an ordinary differential equation of the $(\mu-1)^{th}$ degree between the two variables $\phi^{(\mu-1)}$ and ϕ.

For we have

$$\frac{d\phi'}{d\phi}=\frac{\phi''}{\phi'}, \ldots\ldots \frac{d\phi^{(\mu-2)}}{d\phi}=\frac{\phi^{(\mu-1)}}{\phi'},$$

$$\frac{d\phi^{(\mu-1)}}{d\phi}=\frac{f(\phi, \phi', \ldots.. \phi^{(\mu-1)})}{\phi'}.$$

Differentiating the last equation with respect to ϕ, and attending to the former ones, we shall be able to express $\frac{d^2\phi^{(\mu-1)}}{d\phi^2}$ in terms of the variables $\phi, \phi', \ldots\ldots \phi^{(\mu-1)}$. Proceeding with this in the same way and continuing the process we shall be able to express the series of differential coefficients

$$\frac{d\phi^{(\mu-1)}}{d\phi}, \frac{d^2\phi^{(\mu-1)}}{d\phi^2}, \ldots\ldots \frac{d^{\mu-1}\phi^{(\mu-1)}}{d\phi^{(\mu-1)}}$$

in terms of $\phi, \phi', \ldots.. \phi^{(\mu-1)}$. From these $\mu-1$ equations, eliminating $\phi', \phi'', \ldots.. \phi^{(\mu-2)}$, we shall have a final equation between

$$\phi, \phi^{(\mu-1)}, \frac{d\phi^{(\mu-1)}}{d\phi}, \ldots\ldots \frac{d^{\mu-1}\phi^{(\mu-1)}}{d\phi^{\mu-1}},$$

that is, a differential equation of the $(\mu-1)^{th}$ order between ϕ and $\phi^{(\mu-1)}$.

The complete integral of this equation will be of the form

$$\phi^{(\mu-1)} = f(\phi, c_1, c_2, \ldots\ldots c_{\mu-1}).$$

Differentiating this $\mu-2$ times in succession with respect to ϕ, and continually substituting for the differential coefficients of $\phi^{(\mu-1)}$ their values as before assigned in terms of

$$\phi, \phi', \ldots\ldots \phi^{(\mu-1)},$$

we shall have a system of $\mu-1$ equations connecting the above variables with the constants $c_1, c_2, \ldots\ldots c_{\mu-1}$. Finally, solving these equations with respect to the constants, we shall possess the integrals required in the form

$$F_1(\phi, \phi', \ldots\ldots \phi^{(\mu-1)}) = c_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$F_{\mu-1}(\phi, \phi', \ldots\ldots \phi^{(\mu-1)}) = c_{\mu-1},$$

and each of these will be a common integral of the first two equations of the given system (1).

[On the back of a page of the manuscript the following paragraph occurs, which seems to have been intended as a simplification of the preceding argument which begins with "The complete integral."]

Suppose that a first integral of the equation can be found. Its form will be

$$F\left(\phi, \phi^{(\mu-1)}, \frac{d\phi^{(\mu-1)}}{d\phi}, \ldots\ldots \frac{d^{\mu-2}\phi^{(\mu-1)}}{d\phi^{\mu-2}}\right) = c.$$

Substitute in this for the differential coefficients of $\phi^{(\mu-1)}$ their values before assigned in terms of $\phi, \phi', \phi'', \ldots \phi^{(\mu-1)}$, and we have an integral of the system (3), and therefore a common integral of the first two equations of the system (1).

[We now return to the place at which we inserted a paragraph.]

Just in the same way Jacobi deduces a common integral of the first three equations of the system (1). For representing any one of the first members of the above system by ψ, and deriving thence the new independent integrals $\Delta_3\psi$, $\Delta_3{}^2\psi$,... he substitutes an arbitrary function of these for P in the equation

$$\Delta_3 P = 0.$$

It is evident that the solution of the partial differential equation so found will again be reducible to that of an ordinary differential equation between two variables. And so the process is carried on till all the equations are satisfied.

2. The above remarkable process was developed by Jacobi in connexion with the theory of non-linear partial differential equations of the first order. In that particular connexion it admits of certain reductions tending to diminish the order of the differential equations to be integrated. But these do not affect the general principle of the method. It was in this special form that the theory of the solution of simultaneous linear partial differential equations originated. Jacobi does not consider the theory of equations in which the condition (2) is not satisfied; but the language in which he refers to the condition shews that he had speculated upon the general problem—and it is difficult to conceive that he should have meditated upon it and not arrived at its complete solution.

[The manuscript here gives the first two words of the passage from Jacobi's memoir which is quoted in the *Philosophical Transactions* for 1863, page 486.]

CHAPTER XXVII.

OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

1. IN treating the present subject we shall first consider that class of non-linear partial differential equations of the first order which involves two independent variables, and then proceed to the general theory. The reason for this procedure is that the particular theory, though of course included in the general one, rests upon a somewhat simpler basis, and it was in fact developed by the labours of Lagrange and Charpit long before the general theory was known. The latter we owe to the independent researches of Cauchy and Jacobi.

[Here the manuscript refers to the matter contained in Chap. XIV. Arts. 7 to 12 inclusive; and then passes on to the general theory.]

General Theory.

2. Given an equation of the form

$$z = \phi(x_1, x_2, \dots x_n, a_1, a_2, \dots a_n),$$

the number of arbitrary constants $a_1, a_2, \dots a_n$ involved being equal to the number of the independent variables $x_1, x_2, \dots x_n$, we obtain by differentiation and elimination of the constants a partial differential equation of the first order. Of this the proposed equation is said to constitute a complete primitive.

The form of the above process which it seems best, as throwing light upon the inverse problem of deducing the complete primitive from the partial differential equation, to employ, is the following. Let the given primitive, solved with respect to one of the arbitrary constants a_1, be presented in the form

$$f(x_1, \dots x_n, z, a_2, \dots a_n) = a_1 \text{ ...... (1).}$$

Differentiating with respect to each of the independent variables we have a system of n equations of the forms

$$\left.\begin{array}{l} f_1(x_1, \dots x_n, z, p_1, a_2, \dots a_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ f_n(x_1, \dots x_n, z, p_n, a_2, \dots a_n) = 0 \end{array}\right\} \text{ ...... (2).}$$

These n equations enable us first to eliminate the $n-1$ constants $a_2, \dots\dots a_n$, and so deduce the partial differential equation sought in the form

$$F_1(x_1, \dots x_n, z, p_1, \dots p_n) = 0 \text{ .... (3);}$$

secondly to determine the $n-1$ constants as functions of $x_1, \dots x_n, z, p_1, \dots p_n$ in the forms

$$\left.\begin{array}{l} F_2(x_1, \dots x_n, z, p_1, \dots p_n) = a_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ F_n(x_1, \dots x_n, z, p_1, \dots p_n) = a_n \end{array}\right\} \text{ ...... (4).}$$

As the system formed of these $n-1$ equations, together with the previous one, is merely another form of the system (2) obtained by directly differentiating the primitive, it follows that if from these equations we deduce the values of $p_1, \dots p_n$ as functions of $x_1, \dots x_n, a_2, \dots a_n$, and substitute them in the equation

$$dz = p_1 dx_1 + p_2 dx_2 + \dots + p_n dx_n \text{ .... (5),}$$

they will render that equation integrable, and its integral will be the complete primitive (1), the constant a_1 being regained by integration.

Examining the system (3), (4) we see that the first members of all the equations which it contains are functions of $x_1, \ldots x_n, z, p_1, \ldots p_n$, while the second members are constants. The question then arises, What mutual connexion exists among these functions in virtue of which they yield values of $p_1, \ldots p_n$, which render the equation (5) integrable?

The answer to this question must involve the entire theory of the solution of partial differential equations of the first order, so far as relates to the determination of a complete primitive. Given a partial differential equation of the form (3) it is evident that if we can construct a system of associated equations (4) possessing the character above described, the final value of z obtained by integration of (5) will both satisfy the given equation and contain the requisite number of arbitrary constants. It does not follow from this that it will be the only complete primitive, but it will be a complete primitive.

3. The relation sought is expressed in the following Proposition:

PROPOSITION. *If*

$$F(x_1, \ldots x_n, z, p_1, \ldots p_n) = a,$$
$$\Phi(x_1, \ldots x_n, z, p_1, \ldots p_n) = b$$

represent any two out of a system of n independent equations such that the values of $p_1, \ldots p_n$, thence determined would make the equation

$$dz = p_1 dx_1 + p_2 dx_2 + \ldots + p_n dx_n$$

integrable, then the first members of these equations being represented for simplicity by F and Φ, the condition

$$\Sigma_i \left\{ \left(\frac{dF}{dx_i} + p_i \frac{dF}{dz} \right) \frac{d\Phi}{dp_i} - \frac{dF}{dp_i} \left(\frac{d\Phi}{dx_i} + p_i \frac{d\Phi}{dz} \right) \right\} = 0,$$

the summation extending to all values of i, from 1 *to n inclusive, will be satisfied identically.*

Reciprocally, if the above condition be satisfied identically for each binary combination of functions in the proposed system of equations, and if these functions be independent, then the values of $p_1, \ldots p_n$, *as functions of* $x_1, \ldots x_n, z$, *which they yield, will make the equation*

$$dz = p_1 dx_1 + p_2 dx_2 + \ldots + p_n dx_n$$

integrable.

It will be convenient to begin with the particular case in which the proposed equations do not explicitly contain z, the particular pair to be considered being represented by

$$F(x_1, \ldots x_n, p_1, \ldots p_n) = a,$$

$$\Phi(x_1, \ldots x_n, p_1, \ldots p_n) = b.$$

Differentiating with respect to x_i, and regarding $p_1, \ldots p_n$ as functions of the independent variables, we have

$$\left.\begin{aligned} \frac{dF}{dx_i} + \frac{dF}{dp_1}\frac{dp_1}{dx_i} + \ldots + \frac{dF}{dp_n}\frac{dp_n}{dx_i} = 0, \\ \frac{d\Phi}{dx_i} + \frac{d\Phi}{dp_1}\frac{dp_1}{dx_i} + \ldots + \frac{d\Phi}{dp_n}\frac{dp_n}{dx_i} = 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots (6),$$

to which we may give the form

$$\left.\begin{aligned} \frac{dF}{dx_i} = -\Sigma_j \frac{dF}{dp_j}\frac{dp_j}{dx_i} \\ \frac{d\Phi}{dx_i} = -\Sigma_j \frac{d\Phi}{dp_j}\frac{dp_j}{dx_i} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

the summation with respect to j extending from $j=1$ to $j=n$ inclusive.

From the first of equations (7) multiplied by $\frac{d\Phi}{dp_i}$ subtract the second multiplied by $\frac{dF}{dp_i}$, and sum the result with respect to i from $i=1$ to $i=n$ inclusive. We have

$$\Sigma_i\left(\frac{dF}{dx_i}\frac{d\Phi}{dp_i}-\frac{dF}{dp_i}\frac{d\Phi}{dx_i}\right)$$

$$=-\Sigma_i\Sigma_j\left(\frac{dF}{dp_j}\frac{d\Phi}{dp_i}\frac{dp_j}{dx_i}-\frac{dF}{dp_i}\frac{d\Phi}{dp_j}\frac{dp_j}{dx_i}\right) \text{......} (8).$$

The expression under the double sign of summation in the second member vanishes when $i=j$; we may therefore restrict the summation to unequal values of i and j. Now as for any particular combination of values, e.g. 2, 3, there would exist in the completed member both the terms corresponding to $i=2$, $j=3$, and those corresponding to $j=2$, $i=3$, it is evident that if we employ the symbol Σ_{ij} to denote summation with respect to different combinations of i and j, the second member of the last equation may be expressed in the form

$$\Sigma_{ij}\left(\frac{dF}{dp_j}\frac{d\Phi}{dp_i}\frac{dp_j}{dx_i}-\frac{dF}{dp_i}\frac{d\Phi}{dp_j}\frac{dp_j}{dx_i}\right.$$

$$\left.+\frac{dF}{dp_i}\frac{d\Phi}{dp_j}\frac{dp_i}{dx_j}-\frac{dF}{dp_j}\frac{d\Phi}{dp_i}\frac{dp_i}{dx_j}\right),$$

$$\text{or } \Sigma_{ij}\left\{\left(\frac{dF}{dp_i}\frac{d\Phi}{dp_j}-\frac{dF}{dp_j}\frac{d\Phi}{dp_i}\right)\left(\frac{dp_i}{dx_j}-\frac{dp_j}{dx_i}\right)\right\},$$

so that the equation (8) becomes

$$\Sigma_i\left(\frac{dF}{dx_i}\frac{d\Phi}{dp_i}-\frac{dF}{dp_i}\frac{d\Phi}{dx_i}\right)$$

$$=-\Sigma_{ij}\left\{\left(\frac{dF}{dp_i}\frac{d\Phi}{dp_j}-\frac{dF}{dp_j}\frac{d\Phi}{dp_i}\right)\left(\frac{dp_i}{dx_j}-\frac{dp_j}{dx_i}\right)\right\} \text{......} (9).$$

The number of terms of which the second member expresses the sum is thus $\frac{n(n-1)}{2}$, and it will be observed that as to any particular term it makes no difference in what order the numerical values of i and j are assigned to these quantities; e.g. whether for the combination 2, 3 we make $i=2$, $j=3$, or $i=3$, $j=2$; but we must confine ourselves to one order.

Now when the equation

$$dz = p_1 dx_1 + p_2 dx_2 + \dots + p_n dx_n$$

is integrable in the manner here supposed, we have for all combinations of i and j,

$$\frac{dp_i}{dx_j} = \frac{dp_j}{dx_i}.$$

All the terms in the second member of (9) therefore vanish, and we have

$$\Sigma_i \left(\frac{dF}{dx_i}\frac{d\Phi}{dp_i} - \frac{dF}{dp_i}\frac{d\Phi}{dx_i}\right) = 0.$$

This is the direct form of the Proposition under the particular limitation supposed.

As F, Φ represent, under the same limitation, any two of the first members of the n equations (3), (4), which determine $p_1, \dots p_n$, there will exist $\frac{n(n-1)}{2}$ equations like the above. It is usual to employ for brevity the notation

$$\Sigma_i \left(\frac{dF}{dx_i}\frac{d\Phi}{dp_i} - \frac{dF}{dp_i}\frac{d\Phi}{dx_i}\right) = [F\Phi],$$

and this being done the above system of equations expresses the $\frac{n(n-1)}{2}$ *functions of the form* $[F_i F_j]$ *as linear homogeneous functions of the* $\frac{n(n-1)}{2}$ *quantities of the form* $\frac{dp_i}{dx_j} - \frac{dp_j}{dx_i}$. It is hence that the vanishing of the latter series of quantities secures the vanishing of the former.

The converse truth will therefore be established by shewing that the $\frac{n(n-1)}{2}$ quantities of the form $\frac{dp_i}{dx_j} - \frac{dp_j}{dx_i}$ are, when

$F_1, F_2, \ldots F_n$ are independent with respect to $p_1, p_2, \ldots p_n$, expressible as linear homogeneous functions of the $\frac{n(n-1)}{2}$ functions $[F_iF_j]$.

To avoid complexity of expression I shall establish this for the particular case of $n=3$, and shall shew that the reasoning is general.

The functions F_1, F_2, F_3, being independent with respect to p_1, p_2, p_3, the determinant

$$\begin{vmatrix} \frac{dF_1}{dp_1}, & \frac{dF_1}{dp_2}, & \frac{dF_1}{dp_3} \\ \frac{dF_2}{dp_1}, & \frac{dF_2}{dp_2}, & \frac{dF_2}{dp_3} \\ \frac{dF_3}{dp_1}, & \frac{dF_3}{dp_2}, & \frac{dF_3}{dp_3} \end{vmatrix}$$

does not vanish. This determinant we shall denote by Δ.

In (9) writing for F and Φ first F_2 and F_3, secondly F_3 and F_1, thirdly F_1 and F_2, we have on changing signs the system

$$\left.\begin{aligned} -[F_2F_3] &= \left(\frac{dF_2}{dp_2}\frac{dF_3}{dp_3} - \frac{dF_2}{dp_3}\frac{dF_3}{dp_2}\right)\left(\frac{dp_2}{dx_3} - \frac{dp_3}{dx_2}\right) + \ldots \\ -[F_3F_1] &= \left(\frac{dF_3}{dp_2}\frac{dF_1}{dp_3} - \frac{dF_3}{dp_3}\frac{dF_1}{dp_2}\right)\left(\frac{dp_2}{dx_3} - \frac{dp_3}{dx_2}\right) + \ldots \\ -[F_1F_2] &= \left(\frac{dF_1}{dp_2}\frac{dF_2}{dp_3} - \frac{dF_1}{dp_3}\frac{dF_2}{dp_2}\right)\left(\frac{dp_2}{dx_3} - \frac{dp_3}{dx_2}\right) + \ldots \end{aligned}\right\} \quad (10).$$

Multiply the first equation by $\frac{dF_1}{dp_1}$, the second by $\frac{dF_2}{dp_1}$, the third by $\frac{dF_3}{dp_1}$ and add. Then

$$-\frac{dF_1}{dp_1}[F_2F_3] - \frac{dF_2}{dp_1}[F_3F_1] - \frac{dF_3}{dp_1}[F_1F_2] = \Delta\left(\frac{dp_2}{dx_3} - \frac{dp_3}{dx_2}\right),$$

whence as Δ does not vanish we have, on dividing by it, the function $\frac{dp_2}{dx_3}-\frac{dp_3}{dx_2}$ expressed as a linear homogeneous function of $[F_2F_3]$, $[F_3F_1]$, and $[F_1F_2]$.

In like manner multiplying the equations by $\frac{dF_1}{dp_2}$, $\frac{dF_2}{dp_2}$, $\frac{dF_3}{dp_2}$ respectively, and dividing by Δ, we obtain $\frac{dp_3}{dx_1}-\frac{dp_1}{dx_3}$ as a similar linear homogeneous function, and lastly, multiplying by $\frac{dF_1}{dp_3}$, $\frac{dF_2}{dp_3}$, $\frac{dF_3}{dp_3}$, and proceeding as before, we obtain $\frac{dp_1}{dx_2}-\frac{dp_2}{dx_1}$ as a similar linear homogeneous function.

From all which it follows that when $[F_2F_3]$, $[F_3F_1]$, $[F_1F_2]$ vanish, then

$$\frac{dp_2}{dx_3}-\frac{dp_3}{dx_2},\quad \frac{dp_3}{dx_1}-\frac{dp_1}{dx_3},\quad \frac{dp_1}{dx_2}-\frac{dp_2}{dx_1},$$

will vanish also.

The reasoning is general in its nature. If F_1, F_2, ... F_n are independent with regard to $p_1, p_2, \dots p_n$, the determinant

$$\begin{vmatrix} \frac{dF_1}{dp_1}, \dots\dots \frac{dF_1}{dp_n} \\ \dots\dots\dots\dots \\ \frac{dF_n}{dp_1}, \dots\dots \frac{dF_n}{dp_n} \end{vmatrix} = \Delta \quad \dots\dots\dots\dots (11),$$

does not vanish. This determinant is from its constitution as a determinant linear and homogeneous, not only with respect to any row or column of elements, but also with respect to the possible binary combinations which can be formed of two rows or columns, ternary out of three rows or columns, &c. provided that these combinations are themselves of the form of determinants. In the language of the theory such combinations are called minor determinants. Hence if we construct the system of equations represented by (10), and observe that the coefficients of any particular term of the form $\frac{dp_i}{dx_j}-\frac{dp_j}{dx_i}$ in the several equations form a system of such minors to the general determinant (11), it will be plain that the equations can by multiplication and addition be brought to a form in which the coefficient of that particular term will be Δ. At the same time the coefficients of all the other terms of the form $\frac{dp_i}{dx_j}-\frac{dp_j}{dx_i}$ will vanish. For a little attention will shew that they will be what the determinant Δ would become on making two of its columns or rows of elements equal, and therefore will be identically equal to 0.

Thus the Proposition is generally established for the case in which z does not explicitly appear in the functions

$$F_1, F_2, \ldots\ldots F_n.$$

When z does appear in those functions the equations (6) will be replaced by

$$\frac{dF}{dx_i}+p_i\frac{dF}{dz}+\frac{dF}{dp_1}\frac{dp_1}{dx_i}+\ldots\ldots+\frac{dF}{dp_n}\frac{dp_n}{dx_i}=0,$$

$$\frac{d\Phi}{dx_i}+p_i\frac{d\Phi}{dz}+\frac{d\Phi}{dp_1}\frac{dp_1}{dx_i}+\ldots\ldots+\frac{d\Phi}{dp_n}\frac{dp_n}{dx_i}=0,$$

from which it is seen that the theorem above established will only need to be changed into the form employed in the statement of the general Proposition.

As the above is one of the most important propositions in the entire theory of Differential Equations, it may be desireable to illustrate it by examples.

[There are no examples in the manuscript.]

4. We resume the general theory.

The integration of non-linear partial differential equations may be effected by two distinct methods, both resting upon the ground of the above Proposition. The first of these methods, originally established by a different analysis from that which will here be employed, was discovered by Cauchy (*Exercices d'Analyse*), and rediscovered by Jacobi (*Crelle's Journal*). The second method, discovered by Jacobi at a later period, forms the subject of his posthumous memoir, *Nova Methodus*.....

Cauchy's Method.

We will, as before, begin with the case in which z does not appear explicitly in the proposed partial differential equation, which we shall represent in the form

$$F_1(x_1, \ldots.. x_n, p_1, \ldots.. p_n) = 0 \ldots\ldots\ldots\ldots.. (1).$$

We have seen that to find a complete primitive of the equation it is necessary and sufficient to construct a series of equations

$$\left.\begin{array}{l} F_2(x_1, \ldots.. x_n, p_1, \ldots.. p_n) = a_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_n(x_1 \ldots.. x_n, p_1, \ldots.. p_n) = a_n \end{array}\right\} \ldots\ldots\ldots\ldots (2),$$

such that not only shall the conditions

$$[F_1F_2] = 0, \ldots\ldots [F_1F_n] = 0 \ldots\ldots\ldots\ldots.. (3),$$

connecting the new functions $F_2, \ldots.. F_n$ with F_1, be identically satisfied, but also the series of conditions

$$[F_aF_b] = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

F_a and F_b representing any two of the new functions referred to.

The first of the above series of conditions amounts to this, that $F_2, \ldots.. F_n$ must be integrals of the partial differential equation

$$[F_1P] = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

It is the peculiar aim of Cauchy's method to determine the integrals so as to cause the second series of conditions to be satisfied also. And it is shewn that this will be attained if the integrals of (5), which form the first members of (2), are such that the particular values which $p_1, \ldots. p_{n-1}$ assume when x_n is made to receive any constant value, as 0, are differential coefficients with respect to $x_1, \ldots.. x_{n-1}$ of any single function of those variables, the form of which may be arbitrarily assigned.

The *necessity* of this condition is obvious. If the general values of $p_1, \ldots.. p_n$ are differential coefficients of a function z with respect to $x_1, \ldots.. x_n$, then the particular forms which $p_1, \ldots.. p_{n-1}$ assume when x_n receives any constant value are simply differential coefficients with respect to $x_1, \ldots.. x_{n-1}$ of what z becomes under the same circumstances. To prove its *sufficiency* we must shew that when it is satisfied the conditions represented by (4) will be satisfied also.

Since F_a and F_b are integrals of $[F_1P] = 0$,

$$[F_1F_a] = 0, \quad [F_1F_b] = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Also, since if in (1) and (2) we give to x_n a particular constant value, as 0, and then in (2) regard p_n as a function of

$$x_1, \ldots.. x_{n-1}, \quad p_1, \ldots.. p_{n-1}$$

determined by (1), the system (2) will virtually contain only

$$x_1, \ldots.. x_{n-1}, \quad p_1, \ldots.. p_{n-1},$$

of which $p_1, \ldots.. p_{n-1}$ are differential coefficients of a single function with respect to $x_1, \ldots.. x_{n-1}$, it follows from the proposition of Art. 3, that any two functions F_a and F_b will satisfy mutually the condition

$$\Sigma_{i=1}^{i=n-1} \left(\frac{dF_a}{dx_i}\frac{dF_b}{dp_i} - \frac{dF_a}{dp_i}\frac{dF_b}{dx_i}\right) = 0,$$

the differentiations having reference to

$$x_1, \ldots.. x_{n-1}, \quad p_1, \ldots.. p_{n-1},$$

explicitly as they appear in F_a and F_b, and implicitly as involved in p_n. Thus the developed form of the above equation is

$$\Sigma_{i=1}^{i=n-1}\left\{\left(\frac{dF_a}{dx_i}+\frac{dF_a}{dp_n}\frac{dp_n}{dx_i}\right)\left(\frac{dF_b}{dp_i}+\frac{dF_b}{dp_n}\frac{dp_n}{dp_i}\right)\right.$$
$$\left.-\left(\frac{dF_a}{dp_i}+\frac{dF_a}{dp_n}\frac{dp_n}{dp_i}\right)\left(\frac{dF_b}{dx_i}+\frac{dF_b}{dp_n}\frac{dp_n}{dx_i}\right)\right\}=0,$$

the forms of $\frac{dp_n}{dx_i}$ and $\frac{dp_n}{dp_i}$ being determined from (1).

Performing the multiplications, the above equations will be reduced to the form

$$\Sigma_{i=1}^{i=n-1}\left(\frac{dF_a}{dx_i}\frac{dF_b}{dp_i}-\frac{dF_a}{dp_i}\frac{dF_b}{dx_i}\right)$$
$$+\frac{dF_a}{dp_n}\Sigma_{i=1}^{i=n-1}\left(\frac{dp_n}{dx_i}\frac{dF_b}{dp_i}-\frac{dp_n}{dp_i}\frac{dF_b}{dx_i}\right)$$
$$-\frac{dF_b}{dp_n}\Sigma_{i=1}^{i=n-1}\left(\frac{dp_n}{dx_i}\frac{dF_a}{dp_i}-\frac{dp_n}{dp_i}\frac{dF_a}{dx_i}\right)=0\;\ldots.\;(7).$$

But from the form of the total differential of (1) we see that

$$\frac{dp_n}{dx_i}=-\frac{\frac{dF_1}{dx_i}}{\frac{dF_1}{dp_n}},\quad \frac{dp_n}{dp_i}=-\frac{\frac{dF_1}{dp_i}}{\frac{dF_1}{dp_n}}.$$

Hence

$$\Sigma_{i=1}^{i=n-1}\left(\frac{dp_n}{dx_i}\frac{dF_a}{dp_i}-\frac{dp_n}{dp_i}\frac{dF_a}{dx_i}\right)$$
$$=-\left(\frac{dF_1}{dp_n}\right)^{-1}\Sigma_{i=1}^{i=n-1}\left(\frac{dF_1}{dx_i}\frac{dF_a}{dp_i}-\frac{dF_1}{dp_i}\frac{dF_a}{dx_i}\right).$$

Now since by Art. 3

$$\Sigma_{i=1}^{i=n}\left(\frac{dF_1}{dx_i}\frac{dF_a}{dp_i}-\frac{dF_1}{dp_i}\frac{dF_a}{dx_i}\right)=0,$$

we have

$$\Sigma_{i=1}^{i=n-1}\left(\frac{dF_1}{dx_i}\frac{dF_a}{dp_i}-\frac{dF_1}{dp_i}\frac{dF_a}{dx_i}\right)$$

$$=-\left(\frac{dF_1}{dx_n}\frac{dF_a}{dp_n}-\frac{dF_1}{dp_n}\frac{dF_a}{dx_n}\right);$$

therefore

$$\Sigma_{i=1}^{i=n-1}\left(\frac{dp_n}{dx_i}\frac{dF_a}{dp_i}-\frac{dp_n}{dp_i}\frac{dF_a}{dx_i}\right)$$

$$=\left(\frac{dF_1}{dp_n}\right)^{-1}\left(\frac{dF_1}{dx_n}\frac{dF_a}{dp_n}-\frac{dF_1}{dp_n}\frac{dF_a}{dx_n}\right).$$

In the same way

$$\Sigma_{i=1}^{i=n-1}\left(\frac{dp_n}{dx_i}\frac{dF_b}{dp_i}-\frac{dp_n}{dp_i}\frac{dF_b}{dx_i}\right)$$

$$=\left(\frac{dF_1}{dp_n}\right)^{-1}\left(\frac{dF_1}{dx_n}\frac{dF_b}{dp_n}-\frac{dF_1}{dp_n}\frac{dF_b}{dx_n}\right).$$

The substitution of these values in (7) gives

$$\Sigma_{i=1}^{i=n-1}\left(\frac{dF_a}{dx_i}\frac{dF_b}{dp_i}-\frac{dF_a}{dp_i}\frac{dF_b}{dx_i}\right)$$

$$+\left(\frac{dF_1}{dp_n}\right)^{-1}\left\{\frac{dF_a}{dp_n}\left(\frac{dF_1}{dx_n}\frac{dF_b}{dp_n}-\frac{dF_1}{dp_n}\frac{dF_b}{dx_n}\right)\right.$$

$$\left.-\frac{dF_b}{dp_n}\left(\frac{dF_1}{dx_n}\frac{dF_a}{dp_n}-\frac{dF_1}{dp_n}\frac{dF_a}{dx_n}\right)\right\}=0,$$

or $$\Sigma_{i=1}^{i=n-1}\left(\frac{dF_a}{dx_i}\frac{dF_b}{dp_i}-\frac{dF_a}{dp_i}\frac{dF_b}{dx_i}\right)$$

$$+\frac{dF_a}{dx_n}\frac{dF_b}{dp_n}-\frac{dF_a}{dp_n}\frac{dF_b}{dx_n}=0,$$

or $$\Sigma_{i=1}^{i=n}\left(\frac{dF_a}{dx_i}\frac{dF_b}{dp_i}-\frac{dF_a}{dp_i}\frac{dF_b}{dx_i}\right)=0,$$

which is precisely the equation

$$[F_aF_b]=0.$$

We see therefore that to solve the partial differential equation

$$F_1(x, \ldots.. x_n, p_1, \ldots.. p_n)=0,$$

it is only necessary to construct the linear partial differential equation

$$\Sigma_{i=1}^{i=n}\left(\frac{dF_1}{dx_i}\frac{dP}{dp_i}-\frac{dF_1}{dp_i}\frac{dP}{dx_i}\right)=0,$$

and to obtain $n-1$ independent integrals of this

$$F_2(x_1, \ldots.. x_n, p_1, \ldots.. p_n)=a_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$F_n(x_1, \ldots.. x_n, p_1, \ldots.. p_n)=a_n,$$

such that if we determine from these conjoined with the given equation the values of $p_1, \ldots. p_n$, then those of $p_1, \ldots.. p_{n-1}$ shall, when x_n is made constant, be the partial differential coefficients of one and the same function of $x_1, \ldots.. x_{n-1}$ with respect to these variables in succession.

Now provided that we can find *all* the integrals of the above partial differential equation the particular determination required may be effected in the following manner.

The Lagrangean auxiliary system consists of $2n-1$ ordinary differential equations

$$\frac{dx_1}{-\frac{dF_1}{dp_1}}=\ldots..=\frac{dx_n}{-\frac{dF_1}{dp_n}}=\frac{dp_1}{\frac{dF_1}{dx_1}}=\ldots..=\frac{dp_n}{\frac{dF_1}{dx_n}} \ldots\ldots\ldots (8).$$

These admit of $2n-1$ integrals, one of which will be $F_1 = c_1$; and this will agree with the given equation if we make $c_1 = 0$. We have therefore, besides the particular integral $F_1 = 0$, $2n-2$ integrals of the form

$$\left.\begin{aligned} &\phi_2(x_1, \dots x_n, p_1, \dots p_n) = c_2 \\ &\dots\dots\dots\dots\dots\dots\dots \\ &\phi_{2n-1}(x_1, \dots x_n, p_1, \dots p_n) = c_{2n-1} \end{aligned}\right\} \dots\dots\dots (9).$$

Now suppose it required to find a value of z as a function of $x_1, \dots x_n$, which shall satisfy the given partial differential equation, and shall reduce when $x_n = 0$ (or any numerical constant) to a particular given function of $x_1, \dots x_{n-1}$, which we will represent by $\psi(x_1, \dots x_{n-1})$. Then on the assumption that $x_n = 0$, we have first the given equation

$$z = \psi(x_1, \dots x_{n-1}) \dots\dots\dots\dots\dots\dots\dots (10),$$

secondly the derived equations

$$\left.\begin{aligned} p_1 &= \frac{d\psi(x_1, \dots x_{n-1})}{dx_1} \\ &\dots\dots\dots\dots\dots \\ p_{n-1} &= \frac{d\psi(x_1, \dots x_{n-1})}{dx_{n-1}} \end{aligned}\right\} \dots\dots\dots\dots\dots (11).$$

Make in the $2n-1$ integrals $x_n = 0$, and suppose at the same time $x_1, \dots x_{n-1}, p_1, \dots p_{n-1}$ to receive therein the same values as in the above derived equations. Then from the $3n-2$ particular equations which we thus possess in the two systems united (particular because under the assumption that $x_n = 0$), we can eliminate the $2n-1$ particular values of $x_1, \dots x_{n-1}, p_1 \dots p_{n-1}$, and so obtain $n-1$ equations among the constants. These express the conditions which are necessary and sufficient in order that the values of $p_1, \dots p_{n-1}$ thus derived from the integral equations may, when $x_n = 0$, agree with the values assigned in (11). Accordingly if we substitute in these equations of condition for $c_2, \dots c_{2n-1}$ the general values $\phi_2, \dots \phi_{2n-1}$ we shall obtain $n-1$ equations between $x_1, \dots x_n, p_1, \dots p_n$, which will at once be particular integrals of the system (8), and will possess the property that the values of $p_1, \dots p_{n-1}$ which they in conjunction with $F_1 = 0$ give will when $x_n = 0$ reduce to the values given in (11). Hence these values with that of p_n derived from the same equation will make

$$dz - p_1 dx_1 - \dots\dots - p_n dx_n = 0$$

an exact differential equation. In the integral of this it will only remain to determine the constant so as to make the value of z agree with that given in (10). All the conditions will then be satisfied.

We may collect the results of the above investigation into the following Rule:

To obtain an expression for z as a function of the independent variables $x_1, \dots x_n$, which shall satisfy the partial differential equation

$$F(x_1, \dots x_n, p_1, \dots p_n) = 0,$$

and shall when x_n is made equal to 0 (or to any numerical value) reduce to a given function of $x_1, \dots x_{n-1}$, which we will represent by $\psi(x_1, \dots x_{n-1})$.

RULE. Construct the linear partial differential equation

$$\Sigma_{i=1}^{i=n} \left(\frac{dF}{dx_i}\frac{dP}{dp_i} - \frac{dF}{dp_i}\frac{dP}{dx_i}\right) = 0,$$

and forming its auxiliary Lagrangean system deduce its integrals

$$\phi_2 = c_2, \dots \quad \phi_{2n-1} = c_{2n-1},$$

in addition to the known particular integral $F = 0$.

Between the above integrals and the equations

$$p_1 = \frac{d\psi(x_1, \dots x_{n-1})}{dx_1}, \dots \quad p_{n-1} = \frac{d\psi(x_1, \dots x_{n-1})}{dx_{n-1}},$$

eliminate, after making $x_n = 0$, the quantities

$$x_1, \dots x_{n-1}, p_1, \dots p_{n-1}.$$

In the resulting $n-1$ equations replace

$$c_2 \text{ by } \phi_2, \ldots c_{2n-1} \text{ by } \phi_{2n-1},$$

and we shall have a system of equations which with $F=0$ will determine values of $p_1, \ldots p_n$, which will render

$$dz - p_1 dx_1 - \ldots - p_n dx_n$$

an exact differential. The integration of this will give the integral sought.

In the case in which the given partial differential equation is of the form

$$F(x_1, \ldots x_n, z, p_1, \ldots p_n) = 0,$$

z being contained explicitly, the linear equation to be solved is

$$\Sigma_{i=1}^{i=n} \left\{ \left(\frac{dF}{dx_i} + p_i \frac{dF}{dz} \right) \frac{dP}{dp_i} - \frac{dF}{dp_i} \left(\frac{dP}{dx_i} + p_i \frac{dP}{dz} \right) \right\} = 0,$$

and the argument by which it is shewn that the integrals of this to be employed in conjunction with $F=0$ for the determination of $p_1, \ldots p_n$ need only be so conditioned as to make $p_1, \ldots p_{n-1}$ differential coefficients of one and the same function of $x_1, \ldots x_{n-1}$ when $x_n = 0$ is in character the same as that already developed in the present Article. It is only necessary to substitute in its exposition $\frac{dF}{dx_i} + p_i \frac{dF}{dz}$ for $\frac{dF}{dx_i}$, and so for the other functions.

But as the auxiliary system

$$\frac{dx_1}{-\frac{dF}{dp_1}} = \ldots = \frac{dx_n}{-\frac{dF}{dp_n}} = \frac{dz}{-p_1 \frac{dF}{dp_1} - \ldots - p_n \frac{dF}{dp_n}}$$

$$= \frac{dp_1}{\frac{dF}{dx_1} + p_1 \frac{dF}{dz}} = \ldots = \frac{dp_n}{\frac{dF}{dx_n} + p_n \frac{dF}{dz}} \quad \ldots\ldots (12)$$

virtually includes the equation

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0,$$

the ultimate expression of the Rule will be as follows:

To obtain an expression for z as a function of $x_1, \dots x_n$ which shall satisfy the equation

$$F(x_1, \dots x_n, z, p_1, \dots p_n) = 0,$$

and shall when x_n is made equal to 0 (or to any particular constant value) reduce to a given function $\psi(x_1, \dots x_{n-1})$ of the independent variables $x_1, \dots x_{n-1}$.

RULE. Let

$$\phi_2 = c_2, \dots \phi_{2n} = c_{2n}$$

be the $2n-1$ integrals of the auxiliary system (12) which are additional to the particular integral $F=0$. Make in these $2n$ equations $x_n = 0$ and forming the further equations

$$z = \psi(x_1, \dots x_{n-1}),$$

$$p_1 = \frac{d\psi(x_1, \dots x_{n-1})}{dx_1},$$

$$\dots\dots\dots\dots\dots\dots$$

$$p_{n-1} = \frac{d\psi(x_1, \dots x_{n-1})}{dx_{n-1}},$$

eliminate the $2n$ quantities $x_1, \dots x_{n-1}, z, p_1, \dots p_n$. We thus obtain n equations among the constants $c_2, \dots c_{2n}$.

Substitute in these equations ϕ_2 for $c_2, \dots \phi_{2n}$ for c_{2n}, and we have n equations connecting $x_1, \dots x_n, z, p_1, \dots p_n$, from which with the aid of the given equation $p_1, \dots p_n$ may be eliminated, and there will result a single equation connecting $x_1, \dots x_n$ with z. This is the integral sought.

[It appears from the manuscript that an example was to have been supplied here.]

5. Cauchy's method is evidently a general one. But its generality is not of the same kind as that which belongs to Lagrange's solution of linear partial differential equations. It conducts us, not to a form embracing every possible

solution; but to a system of results from which every possible solution may be derived, by arbitrarily varying the form of the function which expresses the initial state of the dependent variable, that is the value of z when $x_n = 0$, and then performing certain eliminations. To obtain a complete primitive we should only have to assume as the form of z when $x_n = 0$ a function of the variables $x_1, \dots x_{n-1}$ involving n independent constants. The form of this function is arbitrary. Each distinct determination of it under the conditions leads to a distinct complete primitive. The number of such complete primitives is infinite.

There are some most important problems in which the knowledge of a single complete primitive is all that is required. For this purpose the method of Jacobi which we shall now give may be employed.

Jacobi's Last Method.

6. Supposing z to be not explicitly involved in the given partial differential equation

$$F_1(x_1, \dots x_n, p_1, p_n) = 0,$$

which we shall as before represent by $F_1 = 0$, the problem of the discovery of a complete primitive consists in the finding of $n - 1$ equations

$$F_2 = a_2, \dots\dots\dots\dots F_n = a_n,$$

such that between any two functions $F_i F_j$ the relation

$$[F_i F_j] = 0 \dots\dots\dots (1)$$

shall be identically satisfied. The values of $p_1, \dots p_n$ deduced from the equations, by rendering

$$dz - p_1 dx_1 - \dots - p_n dx_n = 0$$

integrable lead us to the complete primitive expressed by its integral.

Now the idea upon which Jacobi's later methods rest is that of directly solving the different systems of linear partial differential equations flowing from the general condition (1), not of solving, as in Cauchy's method, one of those equations and then limiting that solution by conditions which virtually involve the satisfaction of the others.

It is evident that the entire series of $\frac{n(n-1)}{2}$ conditions (1) will be satisfied if we determine F_2 to satisfy the single equation

$$[F_1F_2]=0,$$

then F_3 to satisfy the system of two simultaneous partial differential equations

$$[F_1F_3]=0, \quad [F_2F_3]=0,$$

then F_4 to satisfy the system of three simultaneous partial differential equations

$$[F_1F_4]=0, \quad [F_2F_4]=0, \quad [F_3F_4]=0,$$

and so on, until finally F_n is determined by the solution of the system of $n-1$ partial differential equations

$$[F_1F_n]=0, \quad [F_2F_n]=0, \ldots\ldots [F_{n-1}F_n]=0.$$

Now all these are particular cases of the general problem of determining a function P which shall satisfy simultaneously the equations

$$[F_1P]=0, \quad [F_2P]=0, \ldots\ldots [F_nP]=0. \quad (2)$$

$F_1, F_2, \ldots F_n$ being given functions between each pair of which the equation

$$[F_iF_j]=0$$

is identically satisfied. Here P will represent in succession the series $F_2, F_3, \ldots F_n$.

The given system is one of homogeneous linear partial differential equations. It belongs to the class of systems the general theory of which is discussed in Chap. XXVI. But it is not necessary to apply the theory in its general form. We need only a *single* integral; for a single value of each of the functions F_2, F_3, ... F_n suffices in combination with the given value of F_1 for the determination of a complete primitive. Now it may be shewn that the system is of the class discussed in Chapter XXVI. If expressed symbolically in the form

$$\Delta_1 P = 0, \ \Delta_2 P = 0, \dots \Delta_n P = 0,$$

the condition

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0,$$

will be identically satisfied. Hence Jacobi's method for the treatment of systems of this kind may be applied.

That the system is of the kind asserted is a consequence of the following proposition.

PROPOSITION. If the equations

$$[uP] = 0, \quad [vP] = 0$$

are expressed in the symbolic form

$$\Delta P = 0, \quad \Delta' P = 0,$$

then the derived equation

$$(\Delta\Delta' - \Delta'\Delta) P = 0 \quad \text{.................. (3)},$$

will be equivalent to

$$\Big[[uv] P\Big] = 0.$$

For
$$\Delta = \Sigma_{i=1}^{i=n} \left(\frac{du}{dx_i}\frac{d}{dp_i} - \frac{du}{dp_i}\frac{d}{dx_i}\right),$$

$$\Delta' = \Sigma_{i=1}^{i=n} \left(\frac{dv}{dx_i}\frac{d}{dp_i} - \frac{dv}{dp_i}\frac{d}{dx_i}\right).$$

Hence since $\frac{dv}{dx_j}$ is the coefficient of $\frac{dP}{dp_j}$ in $\Delta' P$, and $\frac{du}{dx_j}$ its coefficient in ΔP, its co efficient in the derivedequation (3) will be (Chap. XXV. Art. 5),

$$\Delta \frac{dv}{dx_j} - \Delta' \frac{du}{dx_j};$$

or $$\Sigma_{i=1}^{i=n} \left(\frac{du}{dx_i} \frac{d^2 v}{dp_i dx_j} - \frac{du}{dp_i} \frac{d^2 v}{dx_i dx_j} - \frac{dv}{dx_i} \frac{d^2 u}{dp_i dx_j} + \frac{dv}{dp_i} \frac{d^2 u}{dx_i dx_j} \right),$$

or $$\frac{d}{dx_j} \Sigma_{i=1}^{i=n} \left(\frac{du}{dx_i} \frac{dv}{dp_i} - \frac{du}{dp_i} \frac{dv}{dx_i} \right),$$

or $$\frac{d}{dx_j} [uv].$$

In like manner the coefficient of $\frac{dP}{dx_j}$ is

$$-\frac{d}{dp_j} [uv].$$

Hence $$(\Delta\Delta' - \Delta'\Delta) P = \Sigma_{j=1}^{j=n} \left(\frac{d[uv]}{dx_j} \frac{dP}{dp_j} - \frac{d[uv]}{dp_j} \frac{dP}{dx_j} \right)$$

$$= \Big[[uv] \, P \Big],$$

whence the Proposition is established.

Applying this to the system (2) we see that any derived equation will be of the form

$$\Big[[F_i F_j] \, P \Big] = 0.$$

But $[F_i F_j] = 0$ by the conditions given; hence the condition $(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0$, is identically satisfied.

The results of Chapter XXVI. being thus directly applicable to the system under consideration, we see that a common integral of the system (2) may be found by a series of alternate processes of integration and derivation. We begin by seeking an integral of the first partial differential equation. By a process of derivation, always possible, followed by the integration of a differential equation between two variables, we arrive at a common integral of the first two partial differential equations. Again, by a process of derivation followed by the solution of a differential equation we obtain a common integral of the first three partial differential equations. And so on, until a common integral of all is obtained.

. . . .

7. Another solution of the above problem has recently been given. Beginning as in Jacobi's method by finding an integral of the first partial differential equation, a process of derivation agreeing in principle with Jacobi's, only more extended, *may* lead us without further integration to a point at which the discovery of a common integral of the entire system will depend only upon the solution of a single differential equation of the first order susceptible of being made integrable by a factor. Failing this, it will enable us to convert the given system of partial differential equations into a new system possessing the same general character, but containing one equation less. Upon this the same process may be tried with a similar final alternative—and so on till the required integral is discovered. (*On the Differential Equations of Dynamics. Philosophical Transactions*, 1863).

CHAPTER XXVIII.

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

[THIS Chapter is a reconstruction on a larger scale of part of Chapter XV. At the end of the Chapter reference will be given to other writings of Professor Boole on the subject here discussed.]

1. The general form of a partial differential equation of the second order is

$$F(x, y, z, p, q, r, s, t) = 0 \text{ ....(1)},$$

where

$$p = \frac{dz}{dx}, \quad q = \frac{dz}{dy}, \quad r = \frac{d^2z}{dx^2}, \quad s = \frac{d^2z}{dx\,dy}, \quad t = \frac{d^2z}{dy^2}.$$

It is only in particular cases that the equation admits of integration, and the most important is that in which the differential coefficients of the second order present themselves only in the first degree; the equation thus assuming the form

$$Rr + Ss + Tt = V \text{ ....(2)},$$

in which R, S, T, and V are functions of x, y, z, p and q.

The most important part of the theory of the solution of this equation is due to Monge, and was extended by Ampère to the more general equation

$$Rr + Ss + Tt + U(s^2 - rt) = V \text{ ....(3)}.$$

This equation, together with the particular equation of Monge, and the equation

$$Rr + Ss + Tt + U(s^2 - rt) = 0,$$

both which though falling under Ampère's general form possess peculiarities demanding special notice, I propose to consider in this Chapter. I shall in conclusion make some observations on the theory of partial differential equations of the second order with more than two independent variables.

Monge's method, and Ampère's in so far as it is an extension of Monge's, consists in a certain procedure for discovering either one or two first integrals of the form

$$u = f(v) \text{.............................} (4),$$

u and v being determinate functions of x, y, z, p, and q; and f being an arbitrary functional symbol. From these first integrals, singly or in combination, the second integral involving two arbitrary functions is obtained by a subsequent integration.

Now this procedure involves the assumption that the proposed equation admits of a first integral of the form (4). But such is not always the case. There exist primitive equations involving two arbitrary functions, from which by proceeding to a second differentiation both functions may be eliminated and an equation of the form (2) obtained, but from which it is impossible to eliminate one function only so as to lead to an intermediate equation of the form (4). Especially this happens if the primitive involve an arbitrary function and its derived function together. Thus the primitive

$$z = \phi(y+x) + \psi(y-x) - x\{\phi'(y+x) - \psi'(y-x)\} \text{...} (5),$$

leads to the partial differential equation of the second order

$$r - t = \frac{2p}{x} \text{.............................} (6),$$

but not through an intermediate equation of the form (4).

It is necessary therefore, not only to consider the case in which the assumed condition is satisfied, but also to notice what has been done in those cases which do not at present fall under the dominion of any known method.

Genesis of the Equation.

2. PROP. I. *A partial differential equation of the first order of the form $u = f(v)$, or its symmetrical equivalent,*

$$F(u, v) = 0,$$

in which u and v are any functions of x, y, z, p, q, always leads to a partial differential equation of the form

$$Rr + Ss + Tt + U(s^2 - rt) = V.$$

For, differentiating the proposed first integral with respect to x, and with respect to y, we have

$$\frac{dF}{du}\left(\frac{du}{dx} + \frac{du}{dz}p + \frac{du}{dp}r + \frac{du}{dq}s\right)$$
$$+ \frac{dF}{dv}\left(\frac{dv}{dx} + \frac{dv}{dz}p + \frac{dv}{dp}r + \frac{dv}{dq}s\right) = 0,$$

$$\frac{dF}{du}\left(\frac{du}{dy} + \frac{du}{dz}q + \frac{du}{dp}s + \frac{du}{dq}t\right)$$
$$+ \frac{dF}{dv}\left(\frac{dv}{dy} + \frac{dv}{dz}q + \frac{dv}{dp}s + \frac{dv}{dq}t\right) = 0.$$

For brevity, write

$$\left(\frac{du}{dx}\right) \text{ for } \frac{du}{dx} + p\frac{du}{dz}, \text{ and } \left(\frac{du}{dy}\right) \text{ for } \frac{du}{dy} + q\frac{du}{dz},$$

and then eliminating

$$\frac{dF}{du}, \quad \frac{dF}{dv},$$

we have

$$\left\{\left(\frac{du}{dx}\right)+\frac{du}{dp}r+\frac{du}{dq}s\right\}\left\{\left(\frac{dv}{dy}\right)+\frac{dv}{dp}s+\frac{dv}{dq}t\right\}$$

$$=\left\{\left(\frac{du}{dy}\right)+\frac{du}{dp}s+\frac{du}{dq}t\right\}\left\{\left(\frac{dv}{dx}\right)+\frac{dv}{dp}r+\frac{dv}{dq}s\right\},$$

which, on effecting the multiplication, gives

$$\left\{\frac{du}{dp}\left(\frac{dv}{dy}\right)-\left(\frac{du}{dy}\right)\frac{dv}{dp}\right\}r$$

$$+\left\{\left(\frac{du}{dx}\right)\frac{dv}{dp}-\frac{du}{dp}\left(\frac{dv}{dx}\right)-\left(\frac{du}{dy}\right)\frac{dv}{dq}+\frac{du}{dq}\left(\frac{dv}{dy}\right)\right\}s$$

$$+\left\{\left(\frac{du}{dx}\right)\frac{dv}{dq}-\frac{du}{dq}\left(\frac{dv}{dx}\right)\right\}t$$

$$+\left(\frac{du}{dq}\frac{dv}{dp}-\frac{du}{dp}\frac{dv}{dq}\right)(s^2-rt)$$

$$=\left(\frac{du}{dy}\right)\left(\frac{dv}{dx}\right)-\left(\frac{du}{dx}\right)\left(\frac{dv}{dy}\right)\ldots\ldots\ldots\ldots(7),$$

a result which, since u and v are by hypothesis given functions of x, y, z, p, q, is seen to be a particular case of the general form (3).

We may hence deduce also the conditions under which particular forms included in the general form (3) arise. Thus, in order that the equation $u=f(v)$ may give rise to a partial differential equation of the second order of Monge's form

$$Rr+Ss+Tt=V,$$

it is necessary that the condition

$$\frac{du}{dq}\frac{dv}{dp}-\frac{du}{dp}\frac{dv}{dq}=0$$

should be identically satisfied. This requires, by Chap. II.

Art. 1, that u and v, considered as functions of p and q, should not be independent.

3. The geometrical relations of the equation (3) are also remarkable. It may in particular be shewn that an equation of this form will be satisfied by the equation of any surface which constitutes the envelope of any system of surfaces formed by the variation of three parameters in subjection to two arbitrary conditions. For let the common equation of the enveloped surfaces be

$$z = f(x, y, a, b, c) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

the parameters a, b, c varying in subjection to the conditions

$$\phi_1(a, b, c) = 0, \qquad \phi_2(a, b, c) = 0,$$

conditions which, determining b and c as functions of a, may be reduced to the form

$$b = \phi(a), \qquad c = \psi(a) \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Now the values of p and q being the same for any point in the envelope as for the same point in the generating surface, we have for all such points

$$p = \frac{df(x, y, a, b, c)}{dx}, \qquad q = \frac{df(x, y, a, b, c)}{dy} \ldots\ldots\ldots (10).$$

These two equations in conjunction with (9) enable us to determine a, b, c as functions of x, y, z, p, q. Let these values be

$$a = u, \quad b = v, \quad c = w.$$

Then substituting in (9) we have

$$v = \phi(u), \quad w = \psi(u),$$

equations which hold for *all* such points. These are then the partial differential equations of the first order of the envelope.

Now each of these equations is of the general form (4); whence by Prop. I. the partial differential equation of the second order is of the form (3), as was to be proved.

Let us actually construct this equation.

Differentiating the first of the equations (10) with respect to x and to y, and regarding therein a as a function of those variables, and b and c as functions of a, we have

$$r = \frac{d^2f}{dx^2} + \left(\frac{d^2f}{dadx} + \frac{d^2f}{dbdx}\frac{db}{da} + \frac{d^2f}{dcdx}\frac{dc}{da}\right)\frac{da}{dx},$$

$$s = \frac{d^2f}{dxdy} + \left(\frac{d^2f}{dadx} + \frac{d^2f}{dbdx}\frac{db}{da} + \frac{d^2f}{dcdx}\frac{dc}{da}\right)\frac{da}{dy},$$

from which we readily derive

$$\left(r - \frac{d^2f}{dx^2}\right)\frac{da}{dy} - \left(s - \frac{d^2f}{dxdy}\right)\frac{da}{dx} = 0.$$

Proceeding in the same way with the second equation of the system (10) we have

$$\left(s - \frac{d^2f}{dxdy}\right)\frac{da}{dy} - \left(t - \frac{d^2f}{dy^2}\right)\frac{da}{dx} = 0.$$

Hence, eliminating $\frac{da}{dx}$ and $\frac{da}{dy}$, we have

$$\left(s - \frac{d^2f}{dxdy}\right)^2 - \left(r - \frac{d^2f}{dx^2}\right)\left(t - \frac{d^2f}{dy^2}\right) = 0$$

or $$\frac{d^2f}{dy^2}r - 2\frac{d^2f}{dxdy}s + \frac{d^2f}{dx^2}t + s^2 - rt = \frac{d^2f}{dx^2}\frac{d^2f}{dy^2} - \left(\frac{d^2f}{dxdy}\right)^2,$$

the equation sought.

Comparing this with the general form (3) we have the equations

$$\frac{\frac{d^2f}{dy^2}}{R} = \frac{-2\frac{d^2f}{dxdy}}{S} = \frac{\frac{d^2f}{dx^2}}{T} = \frac{1}{U} = \frac{\frac{d^2f}{dx^2}\frac{d^2f}{dy^2} - \left(\frac{d^2f}{dxdy}\right)^2}{V},$$

whence eliminating $\frac{d^2f}{dx^2}$, $\frac{d^2f}{dy^2}$, and $\frac{d^2f}{dxdy}$ we arrive at the equation,

$$S^2 + 4(UV - RT) = 0.$$

This then is the condition which must be satisfied in order that the equation (3) may admit of an integral representing the envelope of a system of surfaces in which three parameters vary in subjection to two connecting conditions. It is only proved however to be a *necessary*, not to be a *sufficient*, condition.

Solution of the equation $Rr + Ss + Tt + U(s^2 - rt) = V$, when a first integral of the form $F(u, v) = 0$, exists.

4. In the following sections we propose

1st. To shew that when a first integral of the above form exists, its discovery depends upon the solution of two simultaneous partial differential equations of the first order resolvable into linear equations.

2ndly. To shew how from such first integral or integrals the second integral is to be obtained.

PROP. II. *If the equation*

$$Rr + Ss + Tt + U(s^2 - rt) = V$$

admit of a first integral of the form $F(u, v) = 0$, in which u and v are functions of x, y, z, p, q, then will $F(u, v)$ considered as a function of x, y, z, p, q, and represented as such for brevity by F satisfy the two partial differential equations of the first order,

$$R\left(\frac{dF}{dx}\right)\frac{dF}{dq} + T\left(\frac{dF}{dy}\right)\frac{dF}{dp} + U\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right)$$
$$+ V\frac{dF}{dp}\frac{dF}{dq} = 0,$$

$$R\left(\frac{dF}{dq}\right)^2 - S\frac{dF}{dq}\frac{dF}{dp} + T\left(\frac{dF}{dp}\right)^2$$
$$+ U\left\{\left(\frac{dF}{dx}\right)\frac{dF}{dp} + \left(\frac{dF}{dy}\right)\frac{dF}{dq}\right\} = 0,$$

in which

$$\left(\frac{dF}{dx}\right) = \frac{dF}{dx} + p\frac{dF}{dz}, \quad \left(\frac{dF}{dy}\right) = \frac{dF}{dy} + q\frac{dF}{dz}.$$

Regarding the function F in the proposed integral $F=0$ simply as a function of x, y, z, p, q, we have

$$\left.\begin{aligned}\left(\frac{dF}{dx}\right) + \frac{dF}{dp}r + \frac{dF}{dq}s = 0\\ \left(\frac{dF}{dy}\right) + \frac{dF}{dp}s + \frac{dF}{dq}t = 0\end{aligned}\right\} \text{..........} (11).$$

On the other hand, regarding F as a function of x, y, z, p, q, mediately through u and v, we have the system

$$\left.\begin{aligned}\frac{dF}{du}\left\{\left(\frac{du}{dx}\right) + \frac{du}{dp}r + \frac{du}{dq}s\right\} + \frac{dF}{dv}\left\{\left(\frac{dv}{dx}\right) + \frac{dv}{dp}r + \frac{dv}{dq}s\right\} = 0,\\ \frac{dF}{du}\left\{\left(\frac{du}{dy}\right) + \frac{du}{dp}s + \frac{du}{dq}t\right\} + \frac{dF}{dv}\left\{\left(\frac{dv}{dy}\right) + \frac{dv}{dp}s + \frac{dv}{dq}t\right\} = 0,\end{aligned}\right\} (12),$$

and these systems are *equivalent*.

Now if from the second of these systems we eliminate $\frac{dF}{du}$ and $\frac{dF}{dv}$, we obtain (Art. 2), a result which must be equivalent to the proposed partial differential equation,

$$Rr + Ss + Tt + U(s^2 - rt) = V \text{............} (13).$$

This equation then considered as a relation between r, s, t, must be an algebraical consequence of the relations (12), and therefore of the equations (11). If then we determine algebraically two of the quantities r, s, t, (we select r, t) from the system, and substitute their values in (13), that equation ought to be satisfied independently of the value of the remaining quantity s. Now supposing p and q to be both contained in F, so that neither $\frac{dF}{dp}$ nor $\frac{dF}{dq}$ vanish, we have from (11),

$$r = -\frac{\left(\frac{dF}{dx}\right) + \frac{dF}{dq} s}{\frac{dF}{dp}}, \qquad t = -\frac{\left(\frac{dF}{dy}\right) + \frac{dF}{dp} s}{\frac{dF}{dq}},$$

substituting which in (13) there results

$$R\left(\frac{dF}{dx}\right)\frac{dF}{dq} + T\left(\frac{dF}{dy}\right)\frac{dF}{dp} + U\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right) + V\frac{dF}{dp}\frac{dF}{dq}$$
$$+\left\{R\left(\frac{dF}{dq}\right)^2 - S\frac{dF}{dq}\frac{dF}{dp} + T\left(\frac{dF}{dp}\right)^2\right.$$
$$\left. + U\left(\frac{dF}{dx}\right)\frac{dF}{dp} + U\left(\frac{dF}{dy}\right)\frac{dF}{dq}\right\} s = 0 \;\ldots\ldots (14).$$

Now as this equation is to be satisfied in virtue of the constitution of R, S, T, U, V, and the function F, and independently of s, both the coefficient of s and the absolute term not containing s must be separately equated to 0. Thus F considered as a function of x, y, z, p, q, and containing p, q, at least must satisfy the partial differential equations

$$\left.\begin{array}{l} R\left(\frac{dF}{dx}\right)\frac{dF}{dq} + T\left(\frac{dF}{dy}\right)\frac{dF}{dp} \\ \qquad + U\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right) + V\frac{dF}{dp}\frac{dF}{dq} = 0 \\ R\left(\frac{dF}{dq}\right)^2 - S\frac{dF}{dp}\frac{dF}{dq} + T\left(\frac{dF}{dp}\right)^2 \\ \qquad + U\left\{\left(\frac{dF}{dx}\right)\frac{dF}{dp} + \left(\frac{dF}{dy}\right)\frac{dF}{dq}\right\} = 0 \end{array}\right\} \;\ldots\ldots (15).$$

This result may also be established by forming the equations of condition which express the proportionality of $R, S, \ldots V$, to the corresponding quantities in the constructed equation (7). From these equations of condition it is actually possible to eliminate in two distinct ways the quantities $\left(\frac{dv}{dx}\right), \left(\frac{dv}{dy}\right), \frac{dv}{dp}, \frac{dv}{dq}$, the result being the formation of two partial differential equations for u agreeing in form with those above given for F. (See the memoir *Ueber die partielle Differentialgleichung... Crelle's Journal*, Vol. 61.) The actual transition from the former to the latter rests upon the consideration that the equation $F(u, v) = 0$, when F is arbitrary, is not really less general than the form $\Phi\{F(u, v), v\} = 0$, in which the Φ is arbitrary. And here u has been replaced by $F(u, v)$.

The only condition respecting the application of the above equations is that we do not admit any relations which make either $\frac{dF}{dp}$ or $\frac{dF}{dq}$ to vanish.

5. PROP. III. *The solution of the system of partial differential equations established in the last proposition may in all cases be made to depend upon that of simultaneous linear partial differential equations of the first order.*

In demonstrating this proposition we shall consider first the case in which $U = 0$, then the case in which $V = 0$, lastly the case in which neither of these quantities vanishes. The ground of this division will appear in the investigation.

Case 1. Suppose $U = 0$. The equation then is of Monge's form,

$$Rr + Ss + Tt = V.$$

The second equation of the system (15) becomes

$$R\left(\frac{dF}{dq}\right)^2 - S\frac{dF}{dp}\frac{dF}{dq} + T\left(\frac{dF}{dp}\right)^2 = 0,$$

and therefore breaks up into the equations

$$\frac{dF}{dq} - m_1 \frac{dF}{dp} = 0, \quad \frac{dF}{dq} - m_2 \frac{dF}{dp} = 0,$$

m_1 and m_2 being the roots of the quadratic equation

$$Rm^2 - Sm + T = 0 \quad \text{..................(16).}$$

As each of the above constituent equations is of the form

$$\frac{dF}{dq} = m \frac{dF}{dp},$$

the system (15) may be reduced to the form

$$Rm\left(\frac{dF}{dx}\right)\frac{dF}{dp} + T\left(\frac{dF}{dy}\right)\frac{dF}{dp} + Vm\frac{dF}{dp}\frac{dF}{dp} = 0,$$

which breaks up into the equations

$$\frac{dF}{dp} = 0, \quad Rm\left(\frac{dF}{dx}\right) + T\left(\frac{dF}{dy}\right) + Vm\frac{dF}{dp} = 0.$$

The former of these we must reject (Art. 4). There remains for the determination of F the system of *linear* partial differential equations

$$\left.\begin{aligned} \frac{dF}{dq} - m\frac{dF}{dp} = 0 \\ Rm\left(\frac{dF}{dx}\right) + T\left(\frac{dF}{dy}\right) + Vm\frac{dF}{dp} = 0 \end{aligned}\right\} \text{.........(17),}$$

and there will exist either one or two systems included under this form, according as the roots of the quadratic (16) are equal or unequal.

Case II. Let $V=0$. The system (15) then becomes

$$R\left(\frac{dF}{dx}\right)\frac{dF}{dq}+T\left(\frac{dF}{dy}\right)\frac{dF}{dp}+U\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right)=0,$$

$$R\left(\frac{dF}{dq}\right)^2-S\frac{dF}{dq}\frac{dF}{dp}+T\left(\frac{dF}{dp}\right)^2$$

$$+U\left\{\left(\frac{dF}{dx}\right)\frac{dF}{dp}+\left(\frac{dF}{dy}\right)\frac{dF}{dq}\right\}=0.$$

Eliminate U by multiplying the first equation by

$$\left(\frac{dF}{dx}\right)\frac{dF}{dp}+\left(\frac{dF}{dy}\right)\frac{dF}{dq},$$

the second by

$$\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right),$$

and subtracting; we obtain, after rejection of the common factor $\frac{dF}{dp}\frac{dF}{dq}$,

$$R\left(\frac{dF}{dx}\right)^2+S\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right)+T\left(\frac{dF}{dy}\right)^2=0.$$

We shall put this equation in the place of the second equation of the system. This we are permitted to do under the restriction that in seeking to satisfy the system so changed we do not make use of any relations which would cause either of the two factors employed in the process of elimination to vanish or become infinite.

The new equation reduces to one equation, or breaks up into two equations of the form

$$\left(\frac{dF}{dx}\right)-m\left(\frac{dF}{dy}\right)=0 \quad\text{.....................(18)},$$

m being determined by the quadratic equation

$$Rm^2+Sm+T=0.$$

Making $\left(\frac{dF}{dx}\right) = m\left(\frac{dF}{dy}\right)$ in the first equation of the system (15), we get

$$\left(\frac{dF}{dy}\right)\left\{Rm\frac{dF}{dq} + T\frac{dF}{dp} + Um\left(\frac{dF}{dy}\right)\right\} = 0,$$

which breaks up into

$$\left(\frac{dF}{dy}\right) = 0, \quad Rm\frac{dF}{dq} + T\frac{dF}{dp} + Um\left(\frac{dF}{dy}\right) = 0.$$

But if we combine the first of these with (18), we obtain

$$\left(\frac{dF}{dy}\right) = 0, \quad \left(\frac{dF}{dx}\right) = 0,$$

and this combination causing both the factors employed in the elimination of U to vanish must be rejected. There remains then the combination

$$\left.\begin{aligned} \left(\frac{dF}{dx}\right) - m\left(\frac{dF}{dy}\right) &= 0 \\ Rm\frac{dF}{dq} + T\frac{dF}{dp} + Um\left(\frac{dF}{dy}\right) &= 0 \end{aligned}\right\}\text{............(19)},$$

and this will represent either one or two systems of equations according as the quadratic determining m has equal or unequal roots.

Case III. Let neither $U = 0$ nor $V = 0$.

Multiply the second equation of the system (15) by an indeterminate quantity l, and add to the first; then we have

$$\begin{aligned} Rl\left(\frac{dF}{dq}\right)^2 + Tl\left(\frac{dF}{dp}\right)^2 &+ U\left(\frac{dF}{dx}\right)\left(\frac{dF}{dy}\right) \\ &+ Ul\left(\frac{dF}{dx}\right)\frac{dF}{dp} + T\left(\frac{dF}{dy}\right)\frac{dF}{dp} \\ &+ Ul\left(\frac{dF}{dy}\right)\frac{dF}{dq} + R\left(\frac{dF}{dx}\right)\frac{dF}{dq} \\ &+ (V - Sl)\frac{dF}{dp}\frac{dF}{dq} = 0 \text{.....................(20)}. \end{aligned}$$

We shall enquire whether it is possible so to determine l as to resolve this into linear factors.

We might investigate this by resolving the equation as a quadratic with respect to $\frac{dF}{dq}$ or $\frac{dF}{dp}$. But the form of the equation suggests what the forms of the linear factors must be if the resolution be possible. For as the squares of $\frac{dF}{dq}$ and $\frac{dF}{dp}$ both appear, and these squares alone, in the function to be resolved, it is clear that $\frac{dF}{dq}$ and $\frac{dF}{dp}$ will be the only differential coefficients of F which will appear in both linear factors in common. The most general supposition possible is then that one factor shall contain $\frac{dF}{dq}$ and $\frac{dF}{dp}$ with $\left(\frac{dF}{dx}\right)$, the other the same with $\left(\frac{dF}{dy}\right)$.

Assuming then one factor to be of the form

$$l\frac{dF}{dq}+m\frac{dF}{dp}+n\left(\frac{dF}{dx}\right),$$

it is seen from the form of the coefficients of the first three terms of (20) that the other factor must be of the form

$$R\frac{dF}{dq}+\frac{Tl}{m}\frac{dF}{dp}+\frac{U}{n}\left(\frac{dF}{dy}\right),$$

and the resolved form of (20) must be

$$\left\{R\frac{dF}{dq}+\frac{Tl}{m}\frac{dF}{dp}+\frac{U}{n}\left(\frac{dF}{dy}\right)\right\}\left\{l\frac{dF}{dq}+m\frac{dF}{dp}+n\left(\frac{dF}{dx}\right)\right\}=0.$$

Multiplying out and equating coefficients, we obtain the conditions

$$Ul = \frac{Tln}{m},$$

$$T = \frac{Um}{n},$$

$$Ul = \frac{Ul}{n},$$

$$R = Rn,$$

$$V - Sl = Rm + \frac{Tl^2}{m}.$$

The third and fourth of these conditions are equivalent, and give $n = 1$. The first and second are also equivalent, and give $m = \frac{T}{U}$. These values reduce the last equation of condition to

$$Ul^2 + Sl + \frac{RT}{U} - V = 0,$$

so that l is determined by a quadratic. The resolved form of equation (20) now becomes

$$\left\{R\frac{dF}{dq} + Ul\frac{dF}{dp} + U\left(\frac{dF}{dy}\right)\right\}\left\{l\frac{dF}{dq} + \frac{T}{U}\frac{dF}{dp} + \left(\frac{dF}{dx}\right)\right\} = 0.$$

To these results we may give a somewhat simpler form by making $Ul = m$; not the m used above. We have then as the quadratic for determining m,

$$m^2 + Sm + RT - UV = 0 \text{ .... } (21),$$

and as the resolved form of (20),

$$\left\{R\frac{dF}{dq} + m\frac{dF}{dp} + U\left(\frac{dF}{dy}\right)\right\}\left\{m\frac{dF}{dq} + T\frac{dF}{dp} + U\left(\frac{dF}{dx}\right)\right\} = 0.$$

Let m_1 and m_2 be the values of m. Then we have from the last the two distinct equations

$$\left\{R\frac{dF}{dq}+m_1\frac{dF}{dp}+U\left(\frac{dF}{dy}\right)\right\}\left\{m_1\frac{dF}{dq}+T\frac{dF}{dp}+U\left(\frac{dF}{dx}\right)\right\}=0,$$

$$\left\{R\frac{dF}{dq}+m_2\frac{dF}{dp}+U\left(\frac{dF}{dy}\right)\right\}\left\{m_2\frac{dF}{dq}+T\frac{dF}{dp}+U\left(\frac{dF}{dx}\right)\right\}=0,$$

and it is evident that these will be together equivalent to the equations (15) from which they were derived.

Now to satisfy these equations simultaneously it is necessary that we should equate to 0 one linear factor from each of their first members. If we equate to 0 the first linear factors, we have

$$R\frac{dF}{dq}+m_1\frac{dF}{dp}+U\left(\frac{dF}{dy}\right)=0,$$

$$R\frac{dF}{dq}+m_2\frac{dF}{dp}+U\left(\frac{dF}{dy}\right)=0;$$

whence, by subtraction,

$$(m_1-m_2)\frac{dF}{dp}=0.$$

This combination must therefore be rejected (Art. 4). For the same reason must the combination formed by equating to 0 the second linear factors in the left-hand members of the above two equations be rejected. There remains then only the combinations formed by equating to 0 the first factor of one of these members, and the second of the other.

Thus we should have the combination

$$\left.\begin{aligned} R\frac{dF}{dq}+m_1\frac{dF}{dp}+U\left(\frac{dF}{dy}\right)&=0\\ m_2\frac{dF}{dq}+T\frac{dF}{dp}+U\left(\frac{dF}{dx}\right)&=0\end{aligned}\right\}\text{..........(22)},$$

with the combination which would be obtained from this by interchanging m_1 and m_2.

6. It results from the foregoing investigations that the function F is in all cases to be determined by the solution of two simultaneous linear partial differential equations with *five* independent variables. Now the theory developed in Chapter XXV. shews that the number of integrals of such a system cannot exceed *three*. That theory enables us both to determine what the number of integrals is, and to construct the system of ordinary differential equations, reducible to the exact form, upon which their discovery depends.

We have seen that the knowledge of two integrals $u = a$, $v = b$ of the system enables us to construct a general first integral

$$F(u, v) = 0,$$

of the partial differential equation (3). And the solution of this first integral would lead us to the second integral which is the final object sought. But the direct solving of a partial differential equation of the first order which is not linear and which involves in its actual expression an arbitrary function is difficult, and happily it may be avoided here. The following propositions will enable us to accomplish the virtual solution by a different solution, founded however upon the same general principles.

7. PROP. IV. *The integrals of the respective systems of simultaneous linear partial differential equations upon which the determination of F depends are so related that if from two such respective integrals the values of p and q are determined, they will render the equation*

$$dz = pdx + qdy$$

integrable. And in the particular case in which the two systems become identical, any two integrals of the system stand in the same relation.

For, let Φ be an integral of the system (22), and Ψ an integral of the associated system obtained by interchanging m_1 and m_2 in the case in which these quantities are different. Then Φ satisfies the equations

$$R\frac{d\Phi}{dq}+m_1\frac{d\Phi}{dp}+U\left(\frac{d\Phi}{dy}\right)=0,$$

$$m_2\frac{d\Phi}{dq}+T\frac{d\Phi}{dp}+U\left(\frac{d\Phi}{dx}\right)=0;$$

and Ψ satisfies the equations

$$R\frac{d\Psi}{dq}+m_2\frac{d\Psi}{dp}+U\left(\frac{d\Psi}{dy}\right)=0,$$

$$m_1\frac{d\Psi}{dq}+T\frac{d\Psi}{dp}+U\left(\frac{d\Psi}{dx}\right)=0.$$

But the necessary and sufficient condition in order that the values of p and q derived from the equations $\Phi=0$, $\Psi=0$, may render $dz-pdx-qdy$ integrable, is

$$\left(\frac{d\Phi}{dx}\right)\frac{d\Psi}{dp}-\frac{d\Phi}{dp}\left(\frac{d\Psi}{dx}\right)$$
$$+\left(\frac{d\Phi}{dy}\right)\frac{d\Psi}{dq}-\frac{d\Phi}{dq}\left(\frac{d\Psi}{dy}\right)=0 \;\ldots\ldots\; (23).$$

See Chap. XIV. Art. 11, Equation (36).

Now if from the previous equations we determine the values of

$$\left(\frac{d\Phi}{dx}\right),\ \left(\frac{d\Phi}{dy}\right),\ \left(\frac{d\Psi}{dx}\right),\ \left(\frac{d\Psi}{dy}\right),$$

and substitute them in the above equation of condition it will be identically satisfied.

The determination of $\left(\frac{d\Phi}{dx}\right)$,... from the previous systems requires that U should not vanish. Hence the proposition is established except in the case of $U=0$, which is left doubtful.

To examine this case let us revert to the system (17) which is proper to it. To that system since

$$Rm^2 - Sm + T = 0,$$

whence $$Rm_1m_2 = T,$$

we may give the form

$$\frac{dF}{dq} - m_1\frac{dF}{dp} = 0, \quad \left(\frac{dF}{dx}\right) + m_2\left(\frac{dF}{dy}\right) + \frac{V}{R}\frac{dF}{dp} = 0,$$

or the form obtained from this by interchanging m_1 and m_2.

Substituting in these respective forms Φ and Ψ in succession for F, we find

$$\frac{d\Phi}{dq} = m_1\frac{d\Phi}{dp}, \quad \left(\frac{d\Phi}{dx}\right) = -m_2\left(\frac{d\Phi}{dy}\right) - \frac{V}{R}\frac{d\Phi}{dp},$$

$$\frac{d\Psi}{dq} = m_2\frac{d\Psi}{dp}, \quad \left(\frac{d\Psi}{dx}\right) = -m_1\left(\frac{d\Psi}{dy}\right) - \frac{V}{R}\frac{d\Psi}{dp},$$

and these values substituted in (23) reduce it to an identity. Thus the proposition is established generally.

Lastly, as in the case in which the two roots of the quadratic for determining m are equal, the two systems of partial differential equations for determining Φ and Ψ become one, it follows that if from two integrals of that one system we can deduce values of p and q these values will render the equation

$$dz - pdx - qdy = 0$$

integrable.

8. PROP. V. *When the system of simultaneous linear partial differential equations determining F admits of two integrals $u = a$, $v = b$, it will admit or will not admit of a third integral $w = c$, according as the roots of the quadratic determining m are equal or unequal.*

The system in question, (22), becomes when we divide by U and write for $\left(\frac{dF}{dy}\right)$ and $\left(\frac{dF}{dx}\right)$ their full expressions

$$\frac{dF}{dy} + q\frac{dF}{dz} + \frac{m_1}{U}\frac{dF}{dp} + \frac{R}{U}\frac{dF}{dq} = 0,$$

$$\frac{dF}{dx} + p\frac{dF}{dz} + \frac{T}{U}\frac{dF}{dp} + \frac{m_2}{U}\frac{dF}{dq} = 0,$$

$$\text{or} \quad \Delta_1 F = 0, \quad \Delta_2 F = 0,$$

in which

$$\Delta_1 = \frac{d}{dy} + q\frac{d}{dz} + \frac{m_1}{U}\frac{d}{dp} + \frac{R}{U}\frac{d}{dq},$$

$$\Delta_2 = \frac{d}{dx} + p\frac{d}{dz} + \frac{T}{U}\frac{d}{dp} + \frac{m_2}{U}\frac{d}{dq}.$$

Hence the equation

$$(\Delta_1\Delta_2 - \Delta_2\Delta_1)\,F = 0$$

becomes

$$\frac{m_1 - m_2}{U}\frac{dF}{dz} + \left(\Delta_1\frac{T}{U} - \Delta_2\frac{m_1}{U}\right)\frac{dF}{dp} + \left(\Delta_1\frac{m_2}{U} - \Delta_2\frac{R}{U}\right)\frac{dF}{dq} = 0.$$

In this expression the coefficient of the first term only has been calculated.

Now, by the theory developed in Chap. XXV. in order that the two simultaneous partial differential equations should have their full complement of integrals (three) it is necessary that the above equation should be satisfied identically. This involves three conditions, namely,

$$m_1 - m_2 = 0,$$

$$\Delta_1 \frac{T}{U} - \Delta_2 \frac{m_1}{U} = 0,$$

$$\Delta_1 \frac{m_2}{U} - \Delta_2 \frac{R}{U} = 0,$$

the first of which is the one affirmed in the Proposition to be necessary.

Secondly, it is to be shewn that if this condition be satisfied and if the system of given linear equations admit of two integrals $u - a$, $v = b$, it will admit of a third.

Replacing m_1 and m_2 by m the system becomes

$$\frac{dF}{dy} + q\frac{dF}{dz} + \frac{m}{U}\frac{dF}{dp} + \frac{R}{U}\frac{dF}{dq} = 0,$$

$$\frac{dF}{dx} + p\frac{dF}{dz} + \frac{T}{U}\frac{dF}{dp} + \frac{m}{U}\frac{dF}{dq} = 0.$$

Now if we construct from this the corresponding system of *ordinary* differential equations, we shall find it to be

$$dz - pdx - qdy = 0,$$

$$dp - \frac{T}{U}dx - \frac{m}{U}dy = 0,$$

$$dq - \frac{m}{U}dx - \frac{R}{U}dy = 0.$$

Now it is impossible that the first of these equations should be integrated without a previous determination of p and q as functions of x, y, z, seeing that dx, dy, dz are the three differentials entering into that equation. Such determination can only come from the integration of the second and third equations of the system. But if these equations can be integrated in the forms $u = a$, $v = b$, then u and v being particular values of F satisfying the partial differential equations, it follows from the last Proposition that the values of p and q which they will yield will make the first equation integrable. Hence if the system admits of two integrals it will admit of three; as was to be shewn. On the basis of these Propositions the theory of the second integration rests.

Theory of the Second Integration.

9. First suppose the values of m unequal.

Then $u_1 = a_1$, $v_1 = b_1$ being the two integrals (and we have seen that there cannot be more than two) of one of the systems of linear partial differential equations, and $u_2 = a_2$, $v_2 = b_2$ those of the other, the general first integrals of the given system will be

$$\Phi(u_1, v_1) = 0, \qquad \Psi(u_2, v_2) = 0.$$

The values of p and q determined from these will by Proposition IV. render

$$dz - pdx - qdy = 0$$

integrable, and the integral of this will be the *general* integral of the proposed partial differential equation. For it will involve explicitly or implicitly two arbitrary functions derived from those in the first integrals.

It suffices however, following herein Charpit's method, to combine one general first integral derived from the one system with a particular first integral derived from the other system, e.g. the integrals

$$\Phi(u_1, v_1) = 0, \qquad u_2 = a.$$

The values of p and q hence derived, and employed as before, will lead to a second integral involving one arbitrary function and containing two arbitrary constants. This constitutes a complete primitive from which the general solution will be obtained by converting one of the arbitrary constants into an arbitrary function of the other, and eliminating the latter between the equation and the one derived from it by differentiation with respect to that constant.

Secondly, suppose the values of m equal.

In this case we have but one system of partial differential equations so constituted however that if it admits of two integrals it will admit of three.

Let $u=a$, $v=b$, $w=c$ represent these integrals. Then if from these we eliminate p and q we shall obtain a final integral of the form

$$z=f(x, y, a, b, c),$$

and this constitutes a complete primitive from which we shall deduce the general integral by making $b=\phi(a)$, $c=\psi(a)$, and eliminating a between the equations

$$z=f\{x, y, a, \phi(a), \psi(a)\}$$

$$0=\frac{df\{x, y, a, \phi(a), \psi(a)\}}{da}.$$

To prove this let us combine the general and particular first integrals

$$v=\phi(u), \quad u=a.$$

The values of p and q hence obtained make

$$dz-pdx-qdy=0$$

integrable, and the result can be no other than the remaining integral $w=c$, or rather what this would become on eliminating p and q from it. But since the equations by which this integration are to be effected are equivalent to

$$u=a, \qquad v=\phi(a),$$

w will become a function of x, y, z, a and $\phi(a)$. Also by Charpit's method c is to be treated as a function of a, so that ultimately we have the result above assigned.

We have here supposed U not to vanish. If it do the theory assumes another but simpler form. Let

$$v=f(u), \qquad w=\psi(u)$$

be the two general first integrals. Then, since by the condition at the close of Art. 2, if p be eliminated from these equations q will also disappear, it suffices to eliminate them together in order to obtain the general second integral.

10. Although the cases in which $U=0$ and $V=0$ have in the foregoing sections been treated for simplicity apart, their theory might have been deduced from that of the case in which neither U nor V vanishes.

Thus to deduce the equations for the case of $U=0$ eliminate from the general system (22) $\frac{dF}{dq}$ and $\frac{dF}{dp}$ in succession, and we find

$$(RT-m_1m_2)\frac{dF}{dp}-Um_2\left(\frac{dF}{dy}\right)+UR\left(\frac{dF}{dx}\right)=0,$$

$$(RT-m_1m_2)\frac{dF}{dq}+UT\left(\frac{dF}{dy}\right)-Um_1\left(\frac{dF}{dx}\right)=0.$$

But from (21) $\quad RT-m_1m_2=UV.$

Substituting, and then dividing by U we find

$$V\frac{dF}{dp}-m_2\left(\frac{dF}{dy}\right)+R\left(\frac{dF}{dx}\right)=0,$$

$$V\frac{dF}{dq}+T\left(\frac{dF}{dy}\right)-m_1\left(\frac{dF}{dx}\right)=0,$$

the equation determining m_1, m_2 being

$$m^2+Sm+RT=0.$$

This is equivalent to the results of Art. 5, Case I.

11. We found it necessary (Art. 3) in order that the general partial differential equation of this Chapter should be satisfied by the envelope of a system of surfaces the equations of which contain three parameters varying under two conditions that the relation

$$S^2+4(UV-RT)=0$$

should be satisfied.

It appears from Art. 8 that this is but one of three conditions necessary and together sufficient for this purpose. The formal conditions for every form of ultimate solution consistent with the existence of a general first integral $F(u, v)=0$ can be deduced in the same way.

[In the *Bulletin de l'Académie Impériale des Sciences de St Pétersbourg*, Vol. IV. 1862, there is an article entitled *Considérations sur la recherche des integrales premières des équations différentielles partielles du second ordre, par G. Boldt* (Lu le 7 Juin 1861).

The article occupies pages 198—215 of the volume. Although the name does not quite correspond, I consider that to be a misprint, and I attribute the article to Professor Boole, partly from the nature of the contents, and partly because it is known by his friends that he was engaged at a time corresponding to the date here given in the preparation of a mathematical article in French.

The object of the article is to determine the conditions necessary for the existence of a first integral of the equation

$$R\frac{d^2z}{dx^2}+S\frac{d^2z}{dxdy}+T\frac{d^2z}{dy^2}+W=0,$$

where R, S, T, and W are any functions of x, y, z, $\frac{dz}{dx}$ and $\frac{dz}{dy}$; and also to determine the conditions which must hold in order that Ampère's method of integration may be employed.

In Crelle's Journal, Vol. LXI. there is an article by Professor Boole, entitled *Ueber die partielle Differentialgleichung zweiter Ordnung* $Rr+Ss+Tt+U(s^2-rt)=V$.

The article is dated 1862; it occupies pages 309—333 of the volume.

Among Professor Boole's manuscripts I found a memoir very closely resembling the article in Crelle's Journal; it would appear that the memoir was drawn up with a view to publication in the Transactions of some English Scientific Society, and that this design was afterwards abandoned in favour of the article in Crelle's Journal.

After some hesitation I have resolved to print this memoir. Even if the memoir had been identical with the article in Crelle's Journal it would have been convenient to the English reader to be able to avail himself of the investigations; and the memoir contains remarks which do not occur in the article, and which are interesting in connexion with the history of the subject. There is some repetition of matter which has already been given in Chapter XXVIII.; but I was unwilling to impair the completeness of the memoir by abridgment or omission. Accordingly the memoir forms the next Chapter of the present volume.

In Article 2 of the next Chapter will be found the process to which there is an allusion towards the end of Article 4 of Chapter XXVIII.

It is obvious that the subject of partial differential equations of the second order was much studied by Professor Boole. The chronological order of his writings on the subject appears to be as follows:

1. Chapter XV. of the first edition of his work.

2. The article in the *Bulletin* of St Petersburg.

3. The memoir which forms Chapter XXIX. of the present volume.

4. The article in Crelle's Journal.

5. The Chapter XXVIII. of the present volume.]

CHAPTER XXIX.

ON THE SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION $Rr + Ss + Tt + U(s^2 - rt) = V$, IN WHICH R, S, T, U, V ARE GIVEN FUNCTIONS OF x, y, z, p, q.

1. THE equation, the theory of the solution of which I propose to consider in this paper, is remarkable from its connexion with Geometry. If the equation of a surface contain three constants which vary as parameters in subjection to any two conditions connecting them, the generated envelope will satisfy a partial differential equation of the above form. In other words any envelope of the surface

$$F(x, y, z, a, b, c) = 0$$

formed by the variation of a, b, c in subjection to two connecting conditions

$$\phi_1(a, b, c) = 0, \quad \phi_2(a, b, c) = 0$$

is necessarily an integral of a partial differential equation of the form given above.

Now this theorem is the more important, because it is only when three parameters in the equation of a surface vary in subjection to *two* relations that the envelope possesses, irrespectively of the form of the connecting relations, any *definite* character. If there be but *one* connecting relation it is possible to determine that relation so as to make the envelope assume the form of any surface whatever, and therefore the possible system of envelopes is in such case unlimited. If there be *three* connecting relations the parameters become absolutely constant and no envelope exists.

The partial differential equation

$$Rr + Ss + Tt + U(s^2 - rt) = V$$

is remarkable also as including all the cases in which a partial differential equation of the second order admits a first integral of the form

$$u = f(v),$$

u and v being definite functions of x, y, z, p, q, and $f(v)$ arbitrary in form.

Neither of these statements is sufficiently general to constitute a theory of the genesis of the partial differential equation under consideration, but the second one is more general than the first, and is indeed sufficiently so to serve as the ground of an investigation which connects the solution of the equation in all cases with the satisfaction of a system of simultaneous ordinary differential equations of the first order and degree. And this is the ground upon which the method of the paper will rest. I propose to shew, 1st that the solution of the given equation on the assumption that a first integral of the form $u = f(v)$ exists requires the satisfaction of a system of two partial differential equations of the first order and second degree; 2ndly that this system may be resolved into four systems, each consisting of two partial differential equations of the first order and first degree, two of which systems are irrelevant and the other two relevant; 3rdly that the solution of the two relevant systems ultimately depends on the solution of a system of ordinary differential equations of the first order, and that from these ordinary differential equations the given equation of the second order may be deduced independently of the assumption above mentioned. I shall also discuss the theory of the second integration. And I shall exemplify another method of solution connected by a remarkable law of reciprocity with the above method.

First Investigation.

2. PROP. I. *If $u=f(v)$ be a first integral of the equation*

$$Rr+Ss+Tt+U(s^2-rt)=V \dots\dots\dots\dots (1),$$

then will u and v, considered as functions of x, y, z, p, q, each satisfy two partial differential equations of the form

$$\left.\begin{aligned} &R\left(\frac{du}{dx}\right)^2+S\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+T\left(\frac{du}{dy}\right)^2 \\ &\qquad +V\left\{\frac{du}{dp}\left(\frac{du}{dx}\right)+\frac{du}{dq}\left(\frac{du}{dy}\right)\right\}=0 \\ &R\left(\frac{du}{dx}\right)\frac{du}{dq}+T\left(\frac{du}{dy}\right)\frac{du}{dp} \\ &\qquad +U\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+V\frac{du}{dp}\frac{du}{dq}=0 \end{aligned}\right\} \dots\dots\dots\dots (2),$$

in which $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ stand for $\frac{du}{dx}+p\frac{du}{dz}$, and $\frac{du}{dy}+q\frac{du}{dz}$ respectively.

To demonstrate this proposition we shall form directly the partial differential equation of the second order of which $u=f(v)$ is an integral and, comparing that equation with (1), deduce the conditions for the determination of u and v.

Differentiating $u=f(v)$, first with respect to x and secondly with respect to y, we have

$$\frac{du}{dx}+\frac{du}{dz}\frac{dz}{dx}+\frac{du}{dp}\frac{dp}{dx}+\frac{du}{dq}\frac{dq}{dx}$$
$$=f'(v)\left\{\frac{dv}{dx}+\frac{dv}{dz}\frac{dz}{dx}+\frac{dv}{dp}\frac{dp}{dx}+\frac{dv}{dq}\frac{dq}{dx}\right\},$$

$$\frac{du}{dy}+\frac{du}{dz}\frac{dz}{dy}+\frac{du}{dp}\frac{dp}{dy}+\frac{du}{dq}\frac{dq}{dy}$$
$$=f'(v)\left\{\frac{dv}{dy}+\frac{dv}{dz}\frac{dz}{dy}+\frac{dv}{dp}\frac{dp}{dy}+\frac{dv}{dq}\frac{dq}{dy}\right\},$$

or, if we represent $\frac{du}{dx}+p\frac{du}{dz}$ by $\left(\frac{du}{dx}\right)$, $\frac{du}{dy}+q\frac{du}{dz}$ by $\left(\frac{du}{dy}\right)$, $\frac{dp}{dx}$ by r, $\frac{dp}{dy}$ and $\frac{dq}{dx}$ by s, and $\frac{dq}{dy}$ by t,

$$\left(\frac{du}{dx}\right)+r\frac{du}{dp}+s\frac{du}{dq}=f'(v)\left\{\left(\frac{dv}{dx}\right)+r\frac{dv}{dp}+s\frac{dv}{dq}\right\},$$

$$\left(\frac{du}{dy}\right)+s\frac{du}{dp}+t\frac{du}{dq}=f'(v)\left\{\left(\frac{dv}{dy}\right)+s\frac{dv}{dp}+t\frac{dv}{dq}\right\}.$$

Eliminating $f'(v)$ we arrive at the partial differential equation of the second order,

$$\begin{aligned}&\left\{\frac{du}{dp}\left(\frac{dv}{dy}\right)-\frac{dv}{dp}\left(\frac{du}{dy}\right)\right\}r\\ &+\left\{\frac{du}{dq}\left(\frac{dv}{dy}\right)-\frac{dv}{dq}\left(\frac{du}{dy}\right)+\frac{dv}{dp}\left(\frac{du}{dx}\right)-\frac{du}{dp}\left(\frac{dv}{dx}\right)\right\}s\\ &+\left\{\frac{dv}{dq}\left(\frac{du}{dx}\right)-\frac{du}{dq}\left(\frac{dv}{dx}\right)\right\}t+\left\{\frac{du}{dq}\frac{dv}{dp}-\frac{du}{dp}\frac{dv}{dq}\right\}(s^2-rt)\\ &=\left(\frac{du}{dy}\right)\left(\frac{dv}{dx}\right)-\left(\frac{du}{dx}\right)\left(\frac{dv}{dy}\right)\ldots\ldots\ldots\ldots(3).\end{aligned}$$

It is seen that as respects the mode in which the quantities r, s, t are involved this equation is of the same form as the given equation (1). That it may be equivalent, its coefficients must stand to those of (1) in a common ratio μ. This gives

$$\frac{du}{dp}\left(\frac{dv}{dy}\right)-\frac{dv}{dp}\left(\frac{du}{dy}\right)=\mu R\ldots\ldots\ldots\ldots(a),$$

$$\frac{du}{dq}\left(\frac{dv}{dy}\right)-\frac{dv}{dq}\left(\frac{du}{dy}\right)+\frac{dv}{dp}\left(\frac{du}{dx}\right)-\frac{du}{dp}\left(\frac{dv}{dx}\right)=\mu S\ldots\ldots(b),$$

$$\frac{dv}{dq}\left(\frac{du}{dx}\right)-\frac{du}{dq}\left(\frac{dv}{dx}\right)=\mu T\ldots\ldots\ldots\ldots(c),$$

$$\frac{du}{dq}\frac{dv}{dp}-\frac{du}{dp}\frac{dv}{dq}=\mu U \quad\text{.......................................}(d),$$

$$\left(\frac{du}{dy}\right)\left(\frac{dv}{dx}\right)-\left(\frac{du}{dx}\right)\left(\frac{dv}{dy}\right)=\mu V \quad\text{.........................}(e).$$

As we have here five equations which are homogeneous with respect to the four differential coefficients of v and to μ, it is clear that we can, by the elimination of these quantities, obtain a relation connecting the differential coefficients of u with R, S, T, &c. But the peculiar cyclical form of the functions in the first members of the above system enables us to effect this elimination so as to lead to *two* final equations independent of v and μ.

Thus multiplying (a) by $\left(\frac{du}{dx}\right)\frac{du}{dq}$, (c) by $\left(\frac{du}{dy}\right)\frac{du}{dp}$, (d) by $\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)$, and (e) by $\frac{du}{dp}\frac{du}{dq}$, and adding, we find, on rejecting the common factor μ,

$$R\left(\frac{du}{dx}\right)\frac{du}{dq}+T\left(\frac{du}{dy}\right)\frac{du}{dp}+U\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)$$

$$+V\frac{du}{dp}\frac{du}{dq}=0 \quad\text{..........................}(4).$$

Again, multiplying (a) by $\left(\frac{du}{dx}\right)^2$, (b) by $\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)$, (c) by $\left(\frac{du}{dy}\right)^2$, and (e) by $\left(\frac{du}{dx}\right)\frac{du}{dp}+\left(\frac{du}{dy}\right)\frac{du}{dq}$, adding, and again rejecting the common factor μ, we have

$$R\left(\frac{du}{dx}\right)^2+S\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+T\left(\frac{du}{dy}\right)^2$$

$$+V\left\{\left(\frac{du}{dx}\right)\frac{du}{dp}+\left(\frac{du}{dy}\right)\frac{du}{dq}\right\}=0 \quad\text{...............}(5).$$

Hence, u considered as a function of x, y, z, p, q satisfies the two partial differential equations (4), (5), both which are of the first order and second degree.

As u and v enter symmetrically into the system (a), (b), &c., v will also satisfy two partial differential equations of the same form, viz. the equations

$$\left.\begin{aligned} &R\left(\frac{dv}{dx}\right)\frac{dv}{dq}+T\left(\frac{dv}{dy}\right)\frac{dv}{dp} \\ &\qquad +U\left(\frac{dv}{dx}\right)\left(\frac{dv}{dy}\right)+V\frac{dv}{dp}\frac{dv}{dq}=0 \\ &R\left(\frac{dv}{dx}\right)^2+S\left(\frac{dv}{dx}\right)\left(\frac{dv}{dy}\right)+T\left(\frac{dv}{dy}\right)^2 \\ &\qquad +V\left\{\left(\frac{dv}{dx}\right)\frac{dv}{dp}+\left(\frac{dv}{dy}\right)\frac{dv}{dq}\right\}=0 \end{aligned}\right\}\ldots (6).$$

Further, these two systems of equations constitute the *complete* system of equations resulting from the elimination of μ from the five equations (a), (b), (c), &c.; for in their determination, no factor involving either the differential coefficients of u and v, or the quantities R, S, T, &c. has been rejected directly or indirectly.

I am not aware that the above results of elimination have been noticed before.

3. PROP. II. *The system of partial differential equations above obtained for the determination of u, viz.*

$$\left.\begin{aligned} &R\left(\frac{du}{dx}\right)\frac{du}{dq}+T\left(\frac{du}{dy}\right)\frac{du}{dp} \\ &\qquad +U\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+V\frac{du}{dp}\frac{du}{dq}=0 \\ &R\left(\frac{du}{dx}\right)^2+S\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+T\left(\frac{du}{dy}\right)^2 \\ &\qquad +V\left\{\left(\frac{du}{dx}\right)\frac{du}{dp}+\left(\frac{du}{dy}\right)\frac{du}{dq}\right\}=0 \end{aligned}\right\}\ldots\ldots (7)$$

admits of resolution into four systems, each consisting of two linear partial differential equations of the first order. Of these systems two only are relevant to the solution of the problem.

For, multiplying the second by an indeterminate quantity λ, and adding the result to the first, we have

$$R\lambda\left(\frac{du}{dx}\right)^2+(U+S\lambda)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+T\lambda\left(\frac{du}{dy}\right)^2$$

$$+R\left(\frac{du}{dx}\right)\frac{du}{dq}+V\frac{du}{dp}\frac{du}{dq}+T\left(\frac{du}{dy}\right)\frac{du}{dp}$$

$$+V\lambda\left\{\left(\frac{du}{dx}\right)\frac{du}{dp}+\left(\frac{du}{dy}\right)\frac{du}{dq}\right\}=0\ \ldots\ldots\ldots\ldots\ldots(8).$$

Now let us see if it is possible to determine λ so as to make the first member of the equation resolvable into linear factors. We cannot say *à priori* that such resolution is possible as we should be able to do if that member were homogeneous and of the second degree with respect to *three* instead of with respect to the *four* subject variables

$$\left(\frac{du}{dx}\right),\ \left(\frac{du}{dy}\right),\ \left(\frac{du}{dp}\right),\ \left(\frac{du}{dq}\right).$$

Observing that the squares of $\frac{du}{dp}$ and $\frac{du}{dq}$ are wanting in the first member of (8) while those of $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ appear, we are led to assume as the proposed equivalent of that member an expression of the form

$$\left\{R\left(\frac{du}{dx}\right)+m\left(\frac{du}{dy}\right)+n\frac{du}{dp}\right\}\left\{\lambda\left(\frac{du}{dx}\right)+m'\left(\frac{du}{dy}\right)+n'\frac{du}{dq}\right\}.$$

Multiplying the factors of this expression together and then equating the coefficients with those of the first member of (8) we have

$$Rm' + \lambda m = U + S\lambda \quad \ldots\ldots (a),$$

$$mm' = T\lambda \quad \ldots\ldots (b),$$

$$\lambda n = \lambda V = mn' \quad \ldots\ldots (c),$$

$$Rn' = R \quad \ldots\ldots (d),$$

$$nm' = T \quad \ldots\ldots (e),$$

$$nn' = V \quad \ldots\ldots (f),$$

From (b), (c), (d), we find

$$n = V, \quad n' = 1, \quad m = \lambda V, \quad m' = \frac{T}{V},$$

values which will be found to satisfy (e) and (f) also, and which reduce (a) to the form

$$V^2\lambda^2 - SV\lambda + RT - UV = 0.$$

Supposing λ thus determined, the equation (8) becomes

$$\left\{R\left(\frac{du}{dx}\right) + V\lambda\left(\frac{du}{dy}\right) + V\frac{du}{dp}\right\}\left\{\lambda\left(\frac{du}{dx}\right) + \frac{T}{V}\left(\frac{du}{dy}\right) + \frac{du}{dq}\right\} = 0.$$

The result is a little simplified if we retain m in place of λ.

We thus find as the resolved form of the given equation

$$\left\{R\left(\frac{du}{dx}\right) + m\left(\frac{du}{dy}\right) + V\frac{du}{dp}\right\}\left\{m\left(\frac{du}{dx}\right) + T\left(\frac{du}{dy}\right) + V\frac{du}{dq}\right\} = 0 \ldots (9),$$

m being determined by the quadratic

$$m^2 - Sm + RT - UV = 0.$$

If m_1, m_2 be the values of m thus found, we have

$$\left\{R\left(\frac{du}{dx}\right) + m_1\left(\frac{du}{dy}\right) + V\frac{du}{dp}\right\}\left\{m_1\left(\frac{du}{dx}\right) + T\left(\frac{du}{dy}\right) + V\frac{du}{dq}\right\} = 0,$$

$$\left\{R\left(\frac{du}{dx}\right) + m_2\left(\frac{du}{dy}\right) + V\frac{du}{dp}\right\}\left\{m_2\left(\frac{du}{dx}\right) + T\left(\frac{du}{dy}\right) + V\frac{du}{dq}\right\} = 0,$$

and these two equations are manifestly together equal to the system (7).

Now these equations can only be simultaneously satisfied by equating to 0, one factor in the first member of each; and the different combinations which are thus possible give rise to four binary systems of linear equations. Let us examine these systems separately.

If we simultaneously equate to 0 the two first factors of the left-hand members of the last two equations, we have the systems

$$R\left(\frac{du}{dx}\right)+m_1\left(\frac{du}{dy}\right)+V\frac{du}{dp}=0,$$

$$R\left(\frac{du}{dx}\right)+m_2\left(\frac{du}{dy}\right)+V\frac{du}{dp}=0,$$

a system which, when m_1 and m_2 are different, is reducible to the system

$$R\left(\frac{du}{dx}\right)+V\frac{du}{dp}=0, \quad \left(\frac{du}{dy}\right)=0.$$

It is clear that this cannot lead to a value of u satisfying the given differential equation (1), because it takes no account of the forms of S, U, and T. Indeed if we actually eliminate

$$\left(\frac{du}{dx}\right), \ \left(\frac{du}{dy}\right), \ \frac{du}{dp}, \ \frac{du}{dq}$$

from the above equations by means of the system

$$\left.\begin{aligned}\left(\frac{du}{dx}\right)+\frac{du}{dp}r+\frac{du}{dq}s=0\\ \left(\frac{du}{dy}\right)+\frac{du}{dp}s+\frac{du}{dq}t=0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots(10),$$

(derived from the assumed first integral $u = f(v)$ by making $f(v) = c$, and differentiating the result first with respect to x, then with respect to y), we find as the result

$$Vt + R(s^2 - rt) = 0.$$

Again, if we equate to 0 the two last factors of the right-hand members of (10), we have

$$m_1\left(\frac{du}{dx}\right) + T\left(\frac{du}{dy}\right) + V\frac{du}{dq} = 0,$$

$$m_2\left(\frac{du}{dx}\right) + T\left(\frac{du}{dy}\right) + V\frac{du}{dq} = 0,$$

which, if m_1 and m_2 are different, reduce to

$$T\left(\frac{du}{dy}\right) + V\frac{du}{dq} = 0, \quad \left(\frac{du}{dx}\right) = 0.$$

And it is evident that neither are these equations consistent with the given equation (1), because they take no account of S, U, and R. The equation of the second degree to which they actually lead is

$$Vr + T(s^2 - rt) = 0.$$

There remain then the two systems formed by combining the first factor of each one of the first members with the second factor of the other, viz.

$$\left.\begin{aligned} R\left(\frac{du}{dx}\right) + m_1\left(\frac{du}{dy}\right) + V\frac{du}{dp} = 0 \\ m_2\left(\frac{du}{dx}\right) + T\left(\frac{du}{dy}\right) + V\frac{du}{dq} = 0 \end{aligned}\right\} \text{..........} (11),$$

$$\left.\begin{aligned} R\left(\frac{du}{dx}\right) + m_2\left(\frac{du}{dy}\right) + V\frac{du}{dp} = 0 \\ m_1\left(\frac{du}{dx}\right) + T\left(\frac{du}{dy}\right) + V\frac{du}{dq} = 0 \end{aligned}\right\} \text{..........} (12).$$

That these systems are relevant to the solution of the problem under consideration may be shewn by eliminating from either of them by means of (10) the quantities

$$\left(\frac{du}{dx}\right),\ \left(\frac{du}{dy}\right),\ \frac{du}{dp},\ \frac{du}{dq}.$$

The actual result will be

$$V\{Rr + Ss + Tt + U(s^2 - rt) - V\} = 0 \dots\dots\dots (13),$$

which, except in the particular case of $V = 0$, reduces to the given equation.

More generally, if in the equation

$$u = f(v)$$

u and v are any *distinct* solutions of the system (11), the same result of elimination may be deduced. For v by hypothesis satisfies the equations

$$R\left(\frac{dv}{dx}\right) + m_1\left(\frac{dv}{dy}\right) + V\frac{dv}{dp} = 0,$$

$$m_2\left(\frac{dv}{dx}\right) + T\left(\frac{dv}{dy}\right) + V\frac{dv}{dq} = 0.$$

Subtract these equations multiplied by $f'(v)$ from the corresponding equations of (11), and representing $u - f(v)$ by W, we have

$$R\left(\frac{dW}{dx}\right) + m_1\left(\frac{dW}{dy}\right) + V\frac{dW}{dp} = 0,$$

$$m_2\left(\frac{dW}{dx}\right) + T\left(\frac{dW}{dy}\right) + V\frac{dW}{dq} = 0,$$

which being of the same form as (11) it follows that

$$W = 0 \text{ or } u - f(v) = 0$$

also leads to the partial differential equation of the second order (13).

4. Prop. III. *To reduce the determination of the first integrals of* (1) *to the solution of a system of ordinary differential equations.*

Each of the systems (11), (12) presents u as satisfying simultaneously two linear partial differential equations of the first order.

To deduce the value of u thus conditioned it will obviously suffice to multiply in each system one of the partial differential equations by an indeterminate multiplier λ, to add the result to the other equation so as to form a new equation which will, like those from which it is formed, be linear and of the first order, and which on account of the indeterminate character of λ will be equivalent to the two. From the auxiliary equations which we obtain in the process of solution, λ must be eliminated.

If in this way we combine the equations of the system (11), we have, on arranging the resulting equation according to the differential coefficients of u,

$$(R+\lambda m_2)\frac{du}{dx}+(m_1+\lambda T)\frac{du}{dy}$$

$$+\left\{Rp+m_1q+\lambda(Tq+m_2p)\right\}\frac{du}{dz}$$

$$+V\frac{du}{dp}+\lambda V\frac{du}{dq}=0.$$

Hence we have the auxiliary equations

$$\frac{dx}{R+\lambda m_2}=\frac{dy}{m_1+\lambda T}=\frac{dp}{V}=\frac{dq}{\lambda V}=\frac{dz}{Rp+m_1q+\lambda(Tq+m_2p)},$$

$$du=0,$$

and it is to be remembered that m_1, m_2 are the roots of the equation

$$m^2-Sm+RT-UV=0.$$

Eliminating λ from the first four of the above equations we have

$$\left.\begin{aligned} Udq + m_1dx - Rdy &= 0 \\ Udp + m_2dy - Tdx &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\} \qquad \text{(I)}.$$

This then is the system of ordinary differential equations deduced from (11) upon the integration of which the determination of u will depend.

A similar system, differing from the above only in the mutual transposition of m_1 and m_2, is given by (12), viz.

$$\left.\begin{aligned} Udq + m_2dx - Rdy &= 0 \\ Udp + m_1dy - Tdx &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\} \qquad \text{(II)}.$$

If from either of these systems we can deduce two integrals of the forms

$$u = a, \quad v = b,$$

it is obvious, from what precedes, that

$$u = f(v)$$

will constitute a first integral of the proposed (1), and there being two systems in question, two such first integrals, each involving an arbitrary constant may coexist.

5. PROP. IV. *To deduce the second integral of* (1).

It will be necessary to consider separately the cases in which m_1 and m_2 are equal and unequal.

First let m_1 and m_2 be equal.

Both the systems (I), (II) reduce to a single system which may be expressed in the form

$$\left.\begin{aligned} dp &= \frac{T}{U}dx - \frac{m}{U}dy \\ dq &= -\frac{m}{U}dx + Rdy \\ dz &= pdx + qdy \end{aligned}\right\} \qquad (14).$$

Now, since the condition $\frac{dp}{dy} = \frac{dq}{dx}$ is here satisfied, it is manifest that if from any two integrals of the above system of the forms $u = a$, $v = b$, simultaneous values of p and q be determined, these values will render the third equation of the system integrable, and the effect of its integration will be virtually to determine z as a function of x, y, and three arbitrary constants, viz. a, b, and a constant c introduced in the last integration. Let us represent the result in the form

$$z = \phi(x, y, a, b, c) \text{..................} (15).$$

Now what relation will this result bear to the general solution of the partial differential equation given, to the solution which we should obtain by integrating, not the particular equations $u = a$, $v = b$, but the general first integral $u = f(v)$, which includes them both.

To integrate the equation $u = f(v)$ it suffices to deduce any particular equation involving an arbitrary constant b, which, in conjunction with $u = f(v)$ will render

$$dz - pdx - qdy = 0$$

ntegrable, and to integrate the last equation regarding the arbitrary constant of integration as an arbitrary function of b. The result is a complete primitive in which, by the variation of b as a parameter the general integral is implicitly involved.

Now either of the equations $u = a$, $v = b$ will, in conjunction with $u = f(v)$ determine p and q so as to make

$$dz - pdx - qdy = 0$$

integrable. Take the equation $v = b$, then $u = f(v)$ reduces to

$$u = f(b).$$

Thus, in place of the equations $u = a$, $v = b$, of the previous section, we have

$$u = f(b), \quad v = b$$

for the determination of p and q. The constant c introduced in the final integration becomes also, according to the above theory, a function of b, and the complete primitive is of the form

$$z = \phi \{x, y, b, f(b), \psi(b)\} \ldots\ldots\ldots\ldots (16),$$

while the general integral is found by eliminating b between this equation and its differential with respect to b.

The general integral therefore represents the envelope of the surface represented by (15), a, b, c being parameters subject to any two connecting conditions.

As m_1, m_2 are supposed equal, a necessary condition of the possibility of this species of integration is that

$$S^2 - 4(RT - UV) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

the value of m is $\frac{S}{2}$, and the system (14) reduces to

$$\left.\begin{aligned} Udp + \frac{S}{2}dy - Tdx &= 0 \\ Udq + \frac{S}{2}dx - Rdy &= 0 \\ dz - pdx - qdy &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

We conclude therefore that if (17) *be satisfied and we can from* (18) *deduce a value of z in terms of x, y, and three arbitrary constants, the equation expressing that value will be a complete primitive, and the general integral will be found by making the constants vary in subjection to two arbitrary conditions.*

Ex. Let the given equation be

$$xqr + ypt + xy(s^2 - rt) = pq.$$

Here $R = xq$, $S = 0$, $T = yp$, $U = xy$, $V = pq$.

The condition (17) is satisfied, and (18) becomes

$$xydp - ypdx = 0,$$

$$xydq - xqdy = 0,$$

$$dz - pdx - qdy = 0.$$

From the two first of these we find

$$p = ax, \quad q = by,$$

whence from the third,

$$z = \frac{ax^2}{2} + \frac{by^2}{2} + c.$$

This is the complete primitive, and the general primitive consists of all possible equations derived from this by making a, b, c vary in subjection to two conditions.

Ex. 2. Given

$$(1 + q^2) r - 2pqs + (1 + p^2) t - \frac{s^2 - rt}{(1 + p^2 + q^2)^{\frac{1}{2}}}$$
$$= -(1 + p^2 + q^2)^{\frac{3}{2}}.$$

Here the equation for m reduces to

$$m^2 + 2pqm + p^2q^2 = 0,$$

whence $m = -pq$, and the system (18) gives

$$\frac{dq}{(1 + p^2 + q^2)^{\frac{1}{2}}} + pqdx + (1 + q^2)\, dy = 0,$$

$$\frac{dp}{(1 + p^2 + q^2)^{\frac{1}{2}}} + pqdy + (1 + p^2)\, dx = 0.$$

Subtracting the upper equation multiplied by pq from the lower one multiplied by $1+q^2$, and dividing by $1+p^2+q^2$, we have

$$dx + \frac{(1+q^2)\,dp - pqdq}{(1+p^2+q^2)^{\frac{3}{2}}} = 0,$$

whence

$$x + \frac{p}{\sqrt{(1+p^2+q^2)}} = a.$$

In like manner,

$$y + \frac{q}{\sqrt{(1+p^2+q^2)}} = b.$$

Hence determining p and q,

$$dz = -\frac{(x-a)\,dx + (y-b)\,dy}{\sqrt{\{1-(x-a)^2-(y-b)^2\}}}.$$

Therefore $\qquad (x-a)^2 + (y-b)^2 + (z-c)^2 = 1.$

From this form of the complete primitive it is evident that the general integral will represent all possible tubular surfaces formed by the motion through space of a sphere of constant radius unity.

Secondly, let m_1 and m_2 be unequal.

Then since, in neither of the systems (I) and (II) is the condition $\frac{dp}{dy} = \frac{dq}{dx}$ satisfied, from neither system separately can values of p and q be obtained which make $dz = pdx + qdy$ integrable.

But, as will be shewn, any two integrals obtained, the one from the one system and the other from the other, will give values of p and q which will render $dz = pdx + qdy$ integrable, and the general solution will consist of all possible integrals of the latter equation thus obtained.

Or if the *complete* first integral of either system be combined with any *particular* integral involving an arbitrary constant obtained from the other, the two will furnish values of p and q which render $dz = pdx + qdy$ integrable, and its integral will be a complete primitive involving *one* arbitrary function in its expressed form, *another* in the connexion of its two constants; the general primitive being found in the usual way by making the constants vary as parameters in subjection to a single arbitrary connecting condition.

In fact it may be shewn that if we attempt by the process of Charpit or Lagrange to integrate the partial differential equation of the first order $u = f(v)$, deduced we will here suppose from the system (I), we virtually construct the system (II) in the auxiliary equations upon which the process of solution turns. I have obtained a direct proof of this proposition, but I think it preferable and at the same time sufficient, to direct attention to the prior ground upon which it rests in the relations of the systems of partial differential equations (11), (12) from which the systems of ordinary differential equations (I), (II) are derived.

Let $P=0$ represent any integral of the system (11), and $Q=0$ any integral of the system (12). Then we have

$$-V\frac{dP}{dp} = R\left(\frac{dP}{dx}\right) + m_1\left(\frac{dP}{dy}\right)$$

$$-V\frac{dP}{dq} = m_2\left(\frac{dP}{dx}\right) + T\left(\frac{dP}{dy}\right)$$

$$-V\frac{dQ}{dp} = R\left(\frac{dQ}{dx}\right) + m_2\left(\frac{dQ}{dy}\right)$$

$$-V\frac{dQ}{dq} = m_1\left(\frac{dQ}{dx}\right) + T\left(\frac{dQ}{dy}\right).$$

Hence we deduce

$$-V\left\{\frac{dP}{dp}\left(\frac{dQ}{dx}\right)-\frac{dQ}{dp}\left(\frac{dP}{dx}\right)+\frac{dP}{dq}\left(\frac{dQ}{dy}\right)-\frac{dQ}{dq}\left(\frac{dP}{dy}\right)\right\}$$

$$=\left\{R\left(\frac{dP}{dx}\right)+m_1\left(\frac{dP}{dy}\right)\right\}\left(\frac{dQ}{dx}\right)-\left\{R\left(\frac{dQ}{dx}\right)+m_2\left(\frac{dQ}{dy}\right)\right\}\left(\frac{dP}{dx}\right)$$

$$+\left\{m_2\left(\frac{dP}{dx}\right)+T\left(\frac{dP}{dy}\right)\right\}\left(\frac{dQ}{dy}\right)-\left\{m_1\left(\frac{dQ}{dx}\right)+T\left(\frac{dQ}{dy}\right)\right\}\left(\frac{dP}{dy}\right).$$

The second member of this equation is identically 0. Hence dividing by V we have

$$\frac{dP}{dp}\left(\frac{dQ}{dx}\right)-\frac{dQ}{dp}\left(\frac{dP}{dx}\right)+\frac{dP}{dq}\left(\frac{dQ}{dy}\right)-\frac{dQ}{dq}\left(\frac{dP}{dy}\right)=0. \quad\ldots\ldots (19).$$

But this is the known condition under which the values of p and q deduced from the equations $P=0$, $Q=0$ make $dz=pdx+qdy$ integrable; see Chap. XIV. Art. 13, Equation (36).

We conclude then that if from the systems (I), (II) *we can deduce two corresponding systems of integrals*

$$u_1=a_1,\quad v_1=b_1,$$

$$u_2=a_2,\quad v_2=b_2,$$

then will the first integrals of (1) *be*

$$u_1=f_1(v_1),\quad u_2=f_2(v_2),$$

while the second integral will consist of all possible relations obtained either 1st *by specifying the forms of* f_1, f_2 *and obtaining* p *and* q *as functions of* x *and* y *and integrating* $dz=pdx+qdy$, *or* 2nd*ly, by specifying one of the functions* f_1, f_2, *leaving the other arbitrary, determining* p, q, *integrating* $dz=pdx+qdy$, *and regarding the final constant of integration as an arbitrary parameter.*

Ex. Given $ar + bs + ct + e(s^2 - rt) = h$, the coefficients being constant.

Here $R = a$, $S = b$, $T = c$, $U = e$, $V = h$.

Hence m_1, m_2 are the roots of

$$m^2 - bm + ac - eh = 0,$$

and the systems (I), (II) give

$$\left.\begin{array}{l} edq + m_1 dx - ady = 0 \\ edp + m_2 dy - cdx = 0 \end{array}\right\},$$

$$\left.\begin{array}{l} edq + m_2 dx - ady = 0 \\ edp + m_1 dy - cdx = 0 \end{array}\right\}.$$

Whence the first integrals are

$$eq + m_1 x - ay = f_1(ep + m_2 y - cx),$$

$$eq + m_2 x - ay = f_2(ep + m_1 y - cx),$$

from which all possible second integrals are to be derived in the modes above explained.

Let us take the second of those modes and give to the second of the above first integrals the particular form

$$ep + m_1 y - cx = C,$$

C being an arbitrary constant. From this, and from the other integral, left in its complete form, we have

$$p = \frac{cx - m_1 y + C}{e}, \quad q = \frac{ay - m_1 x + f_1\{(m_2 - m_1) y + C\}}{e},$$

whence, substituting in the formula $dz = pdx + qdy$, integrating, replacing the arbitrary form

$$\int f_1(t)\, dt \text{ by } (m_2 - m_1)\, \phi(t),$$

and introducing an arbitrary function of C for the arbitrary constant, we have

$$z=\frac{1}{e}\left[\frac{cx^2}{2}-m_1xy+\frac{ay^2}{2}+Cx+\phi\{(m_2-m_1)y+C\}\right]+\psi(C)$$

for a complete primitive. The general primitive consists of all possible relations obtained by eliminating C between the above equation and

$$0=\frac{1}{e}[x+\phi'\{(m_2-m_1)y+C\}]+\psi'(C),$$

when the forms of ϕ and ψ are specified.

Second Investigation.

6. If from the equation

$$Rr+Ss+Tt+U(s^2-rt)=V \quad\text{..............}\quad (20),$$

we eliminate r and t by means of the equations

$$dp=rdx+sdy,$$

$$dq=sdx+tdy,$$

the result will be

$$[Rdy^2-Sdxdy+Tdx^2-U(dpdx+dqdy)]\,s$$
$$=Rdpdy+Tdqdx-Udpdq-Vdxdy \quad\text{......}\quad (21).$$

There are different considerations (all of them however involving, as I have been led to think, a more or less explicit reference to some theory of the genesis of the given partial differential equation) which indicate that its solution depends upon that of the equations obtained by equating to 0 the part affected and the part not affected by s, viz. upon the solution of the equations

$$Rdy^2-Sdxdy+Tdx^2-U(dpdx+dqdy)=0 \quad\text{......}\quad (22),$$

$$Rdpdy+Tdqdx-Udpdq-Vdxdy=0 \quad\text{............}\quad (23).$$

Without entering into these considerations let us inquire what consequences may be deduced from these equations assumed to be true.

It is seen that these equations are connected by a remarkable reciprocity with the partial differential equations (7). They will in fact be converted into these equations if we change

$$dx,\ dy,\ dp,\ dq,\ U,\ V,\ S \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (24),$$

into

$$-\frac{du}{dy},\ -\frac{du}{dx},\ \frac{du}{dq},\ \frac{du}{dp},\ V,\ U,\ -S \quad\ldots\ldots\ldots\ldots\ (25)$$

respectively. From this formal connexion it follows that if we multiply (22) by λ and add to the result (23), we shall be able to determine λ so as to permit the resolution of the equation thus formed into linear factors. Ultimately we shall, as appears from Art. 3, reduce the system (22), (23) to an equivalent system of the form

$$(-Rdy - m_1dx + Udq)\,(-m_1dy - Tdx + Udp) = 0,$$
$$(-Rdy - m_2dx + Udq)\,(-m_2dy - Tdx + Udp) = 0,$$

m_1 and m_2 being determined by the equation

$$m^2 + Sm + RT - UV = 0,$$

or, changing the sign of m,

$$\left.\begin{array}{l}(-Rdy + m_1dx + Udq)\,(m_1dy - Tdx + Udp) = 0\\ (-Rdy + m_2dx + Udq)\,(m_2dy - Tdx + Udp) = 0\end{array}\right\}\ldots\ (26),$$

m_1 and m_2 being as in the former investigation roots of

$$m^2 - Sm + RT - UV = 0.$$

Equating to 0 the *corresponding* factors of the first members we have

$$\left.\begin{array}{l}-Rdy + m_1dx + Udq = 0\\ -Rdy + m_2dx + Udq = 0\end{array}\right\},$$

$$\left.\begin{array}{l}m_1dy - Tdx + Udp = 0\\ m_2dy - Tdx + Udp = 0\end{array}\right\}.$$

The first of these, m_1 and m_2 being different, is resolvable into

$$Udq - Rdy = 0, \quad dx = 0;$$

the second into

$$Udp - Tdx = 0, \quad dy = 0,$$

and it is obvious that neither of these can lead to the given partial differential equation (1). The first of them combined with the equations

$$dp = rdx + sdy, \quad dq = sdx + tdy \text{ .... } (27),$$

leads in fact to the partial differential equation

$$R - Ut = 0 \text{ .... } (28),$$

the second in like manner leads to

$$T - Ur = 0 \text{ .... } (29).$$

But equating to 0 the *non-corresponding* factors of the first members of (26) we have

$$\left.\begin{aligned} -Rdy + m_1dx + Udq &= 0 \\ m_2dy - Tdx + Udp &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} m_1dy - Tdx + Udp &= 0 \\ -Rdy + m_2dx + Udq &= 0 \end{aligned}\right\}.$$

Now these systems when completed by the equation $dz = pdx + qdy$ agree with the systems (I), (II) deduced in the previous investigation.

It remains to shew that these systems actually lead to the given partial differential equation (1) *directly*. Eliminating from either of them, combined with the system (27) the differentials dx, dy, dp, dq, we shall have as the result

$$U\{Rr + Ss + Tt + U(s^2 - rt) - V\} = 0 \text{ .... } (30),$$

which, rejecting the factor U, as from (13) we rejected V, is the differential equation proposed.

Ground of the Reciprocity above noticed.

7. The reciprocity above noticed is not of a primary character, but is founded upon two prior laws which I shall proceed to demonstrate.

If from the partial differential equations of the system (7) we eliminate V and substitute the resulting equation in the place of the first equation of the system we shall obtain the equivalent system

$$\left.\begin{aligned} R\left(\frac{du}{dq}\right)^2 - S\frac{du}{dq}\frac{du}{dp} + T\left(\frac{du}{dp}\right)^2 & \\ + U\left\{\left(\frac{du}{dx}\right)\frac{du}{dp} + \left(\frac{du}{dy}\right)\frac{du}{dq}\right\} &= 0 \\ R\left(\frac{du}{dx}\right)^2 + S\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) + T\left(\frac{du}{dy}\right)^2 & \\ + V\left\{\left(\frac{du}{dx}\right)\frac{du}{dp} + \left(\frac{du}{dy}\right)\frac{du}{dq}\right\} &= 0 \end{aligned}\right\} \dots(31).$$

These equations are both symmetrical and it will be observed that they are convertible the one into the other by changing

$$\frac{du}{dq},\ \frac{du}{dp},\ \frac{du}{dx},\ \frac{du}{dy},\ U,\ S,\ V \dots\dots\dots\dots (32),$$

into

$$\frac{du}{dx},\ \frac{du}{dy},\ \frac{du}{dq},\ \frac{du}{dp},\ V,\ -S,\ U \dots\dots\dots (33),$$

respectively. This is a law of reciprocity which connects solely the differential coefficients of u and the coefficients U, S, V of the original equation.

Again $u=0$ is by hypothesis a solution of the given partial differential equation. Regarding it however simply as an equation which is *true* and the truth of which is consistent with that of the equations

$$\left.\begin{aligned} dp &= rdx + sdy \\ dq &= sdx + tdy \end{aligned}\right\} \text{..................... (34)},$$

and differentiating it first with respect to x, secondly with respect to y, we have

$$\frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} + \frac{du}{dp}\frac{dp}{dx} + \frac{du}{dq}\frac{dq}{dx} = 0,$$

$$\frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} + \frac{du}{dp}\frac{dp}{dy} + \frac{du}{dq}\frac{dq}{dy} = 0,$$

equations to which we may give the form

$$-\left(\frac{du}{dx}\right) = r\frac{du}{dp} + s\frac{du}{dq}$$

$$-\left(\frac{du}{dy}\right) = s\frac{du}{dp} + t\frac{du}{dq}.$$

Now this system is of the same form as the system (27) and will agree with it if we change

$$-\frac{du}{dx}, -\frac{du}{dy}, \frac{du}{dp}, \frac{du}{dq} \text{ .... (35)},$$

into

$$dp,\ dq,\ dx,\ dy \text{ .... (36)},$$

respectively—a change which does not affect the coefficients of the given equation, and which is therefore the expression of a law of reciprocity distinct from that last noted. The combination of these two laws does however lead to the law exemplified in the researches of the previous Article; see (24) and (25).

The question here arises whether it would not have been better to employ from the first the symmetrical forms (31) o the partial differential equations of the first order and second degree upon which u depends, than the unsymmetrical forms (7). It was indeed from the symmetrical forms that the chief results of this paper were originally obtained, but the unsymmetrical forms lead to the same end in a simpler way, and therefore they have been made use of in the present memoir.

It may be proper to notice, in concluding this section, that the symmetrical forms in ordinary differentials would have emerged in place of the unsymmetrical ones of (22) and (23), if the quantity $s^2 - rt$ had been retained instead of s. The equations

$$dp = rdx + sdy, \quad dq = sdx + tdy,$$

enable us in fact to reduce the given equation (20) to the form

$$Rdp^2 + Sdpdq + Tdq^2 - V(dpdx + dqdy)$$
$$= (s^2 - rt)\{Rdy^2 - Sdxdy + Tdx^2 - U(dpdx + dqdy)\}.$$

Hence arises the symmetrical system

$$Rdp^2 + Sdpdq + Tdq^2 - V(dpdx + dqdy) = 0,$$

$$Rdy^2 - Sdxdy + Tdx^2 - U(dpdx + dqdy) = 0,$$

which is connected with the system (31) by the single law of reciprocity expressed in (35) and (36).

Postscript.

8. At the time when the above investigations engaged my attention I was totally unaware that the subject of them had been discussed by Ampère (*Journal de l'Ecole Polytechnique*, Tom. XI.) and recently by Professor De Morgan (*Cambridge Philosophical Transactions*, Vol. IX. Pt. IV.). I feel it therefore incumbent upon me to state why after acquainting my-

self with the results of their labours, I offer this paper for publication.

The method of Professor de Morgan so far resembles the first method of this paper, and that of Ampère the second, that while the former makes the solution of the problem depend directly upon that of simultaneous partial differential equations of the first order, the latter makes it to depend directly upon the solution of simultaneous ordinary differential equations of the first order. The formal connexion of these methods by the law of reciprocity is, I believe, established for the first time in this paper. The system of partial differential equations of the second degree (7) has not, so far as I am aware, been given before.

But a point which I think of deep importance is the following. By connecting, as in this paper, the differential equations of the second degree, whether ordinary or partial, by an indefinite multiplier which is afterwards determined so as to admit of the resolution of the system into its component linear elements, we assure ourselves that each step of the solution offers a complete sequence to that which has gone before, and it only remains then to separate the different elements and determine whether they are relevant or irrelevant to the end in view. That any such distinction exists has not, so far as I am aware, been noticed before. And it seems to me the more important that it should be noticed because the solution of partial differential equations in cases far more general than those above considered seems to depend upon the satisfaction of simultaneous differential equations of a degree higher than the first. I have in fact by an application of the Calculus of Variations arrived at the conclusion that the theory of the solution of all partial differential equations of the second order, whatever the number of variables may be, is very intimately connected with the satisfaction of a system of differential equations of the type

$$\frac{dF}{dr}dy^2 - \frac{dF}{ds}dxdy + \frac{dF}{dt}dx^2 = 0,$$

$F = 0$ representing the given partial differential equation, x and y any two of the independent variables, and r, s, t the second differential coefficients of the dependent variable with respect to x and y.

I may perhaps at some future day resume the subject, together with an inquiry into the theory of the solution of the partial differential equation of this paper, when the conditions under which the auxiliary equations (I), (II) are supposed to be integrable are not satisfied.

9. NOTE. It may be desirable to establish directly the converse form of one of the results of Proposition IV. For this object we shall shew that the equation of the envelope of

$$z = \phi(x, y, a, b, c) \qquad (1),$$

where a, b, c are connected by any two conditions of the forms

$$\psi(a, b, c) = 0, \quad \chi(a, b, c) = 0,$$

will satisfy a partial differential equation of the form

$$Rr + Ss + Tt + U(s^2 - rt) = V \qquad (2),$$

in which also

$$S^2 = 4(RT - UV).$$

Differentiating (1) we have

$$\left.\begin{aligned} p &= \frac{d\phi}{dx} + \frac{d\phi}{da}\frac{da}{dx} + \frac{d\phi}{db}\frac{db}{dx} + \frac{d\phi}{dc}\frac{dc}{dx} \\ q &= \frac{d\phi}{dy} + \frac{d\phi}{da}\frac{da}{dy} + \frac{d\phi}{db}\frac{db}{dy} + \frac{d\phi}{dc}\frac{dc}{dy} \end{aligned}\right\}$$

and by the nature of an envelope these reduce to

$$p = \frac{d\phi}{dx}, \quad q = \frac{d\phi}{dy} \qquad (3).$$

Again differentiating these equations with respect to x and y, and writing for simplicity

$$\frac{d^2\phi}{dadx} = A, \quad \frac{d^2\phi}{dbdx} = B, \quad \frac{d^2\phi}{dcdx} = C,$$

$$\frac{d^2\phi}{dady} = A', \quad \frac{d^2\phi}{dbdy} = B', \quad \frac{d^2\phi}{dcdy} = C',$$

we have

$$r=\frac{d^2\phi}{dx^2}+A\frac{da}{dx}+B\frac{db}{dx}+C\frac{dc}{dx},$$

$$s=\frac{d^2\phi}{dxdy}+A\frac{da}{dy}+B\frac{db}{dy}+C\frac{dc}{dy},$$

$$s=\frac{d^2\phi}{dxdy}+A'\frac{da}{dx}+B'\frac{db}{dx}+C'\frac{dc}{dx},$$

$$t=\frac{d^2\phi}{dy^2}+A'\frac{da}{dy}+B'\frac{db}{dy}+C'\frac{dc}{dy}.$$

Hence we find

$$\left(s-\frac{d^2\phi}{dxdy}\right)^2-\left(r-\frac{d^2\phi}{dx^2}\right)\left(t-\frac{d^2\phi}{dy^2}\right)$$

$$=\left(A\frac{da}{dy}+B\frac{db}{dy}+C\frac{dc}{dy}\right)\left(A'\frac{da}{dx}+B'\frac{db}{dx}+C'\frac{dc}{dx}\right)$$

$$-\left(A\frac{da}{dx}+B\frac{db}{dx}+C\frac{dc}{dx}\right)\left(A'\frac{da}{dy}+B'\frac{db}{dy}+C'\frac{dc}{dy}\right)$$

$$=(AB'-A'B)\left(\frac{da}{dy}\frac{db}{dx}-\frac{da}{dx}\frac{db}{dy}\right)$$

$$+(BC'-B'C)\left(\frac{db}{dy}\frac{dc}{dx}-\frac{db}{dx}\frac{dc}{dy}\right)$$

$$+(CA'-C'A)\left(\frac{dc}{dy}\frac{da}{dx}-\frac{dc}{dx}\frac{da}{dy}\right).$$

Now since a, b, c are connected by two conditions, so that b and c are functions of x and y only as being functions of a, we have

$$\frac{da}{dy}\frac{db}{dx}-\frac{da}{dx}\frac{db}{dy}=0,\quad \frac{db}{dy}\frac{dc}{dx}-\frac{db}{dx}\frac{dc}{dy}=0,$$

$$\frac{dc}{dy}\frac{da}{dx}-\frac{dc}{dx}\frac{da}{dy}=0.$$

Thus the above equation reduces to

$$\left(s-\frac{d^2\phi}{dxdy}\right)^2-\left(r-\frac{d^2\phi}{dx^2}\right)\left(t-\frac{d^2\phi}{dy^2}\right)=0,$$

or

$$\frac{d^2\phi}{dy^2}r-2\frac{d^2\phi}{dxdy}s+\frac{d^2\phi}{dx^2}t+s^2-rt=\frac{d^2\phi}{dx^2}\frac{d^2\phi}{dy^2}-\left(\frac{d^2\phi}{dxdy}\right)^2 \dots (4).$$

This equation is of the general form (2). Its coefficients $\frac{d^2\phi}{dy^2}$, &c. are determinable as functions of x, y, z, p, q when the form of the complete primitive (1) is given. For this purpose the complete primitive with the two derived equations (3) suffice.

Again, comparing (4) with (2) we have as the conditions of their equivalence

$$\frac{R}{\frac{d^2\phi}{dy^2}}=\frac{S}{-2\frac{d^2\phi}{dxdy}}=\frac{T}{\frac{d^2\phi}{dx^2}}=U=\frac{V}{\frac{d^2\phi}{dx^2}\frac{d^2\phi}{dy^2}-\left(\frac{d^2\phi}{dxdy}\right)^2},$$

conditions which suppose R, S, T, U, V connected by the relation

$$S^2-4(RT-UV)=0.$$

CHAPTER XXX.

ADDITIONS TO CHAPTER XVII.

[THE present Chapter consists of additions to Chapter XVII. Art. 1 was intended to follow Chap. XVII. Art. 1.]

1. The theory of the solution of linear differential equations in a series flows very beautifully from their symbolical expression. It is usual in treating this subject to assume the form of the series, and deduce from the differential equation the law of its coefficients; but the symbolical form of the differential equation determines in reality the form of the solution as well as the law of derivation of its successive terms.

Let us begin with the binomial equation

$$f_0(D)\, u - f_1(D)\, \epsilon^{r\theta} u = 0.$$

Operating on both sides with $\{f_0(D)\}^{-1}$, we have

$$u - \phi(D)\, \epsilon^{r\theta} u = \{f_0(D)\}^{-1} 0,$$

in which $$\phi(D) = \frac{f_1(D)}{f_0(D)}.$$

Hence $$\{1 - \phi(D)\, \epsilon^{r\theta}\} u = \{f_0(D)\}^{-1} 0.$$

Now $\{f_0(D)\}^{-1} 0$ will be determined by the solution of a linear differential equation with constant coefficients, and will be necessarily of the form

$$AP + BQ + CR + \ldots,$$

in which A, B, C, ... are arbitrary constants, and P, Q, R, ... are functions of the independent variable.

We have then

$$\{1-\phi(D)\epsilon^{r\theta}\}u = AP + BQ + CR + \ldots,$$

therefore $u = \{1-\phi(D)\epsilon^{r\theta}\}^{-1}(AP+BQ+CR+\ldots)$.

Now let us represent $\phi(D)\epsilon^{r\theta}$ by ρ; then

$$\begin{aligned} u &= (1-\rho)^{-1}(AP+BQ+CR+\ldots) \\ &= (1+\rho+\rho^2+\rho^3+\ldots)(AP+BQ+CR+\ldots) \\ &= A(1+\rho+\rho^2+\rho^3+\ldots)P \\ &\quad + B(1+\rho+\rho^2+\rho^3+\ldots)Q \\ &\quad + C(1+\rho+\rho^2+\rho^3+\ldots)R \\ &\quad + \ldots \end{aligned}$$

Represent the first line of the above expression by u_1, then since

$$\begin{aligned} \rho^m &= \phi(D)\epsilon^{r\theta}\phi(D)\epsilon^{r\theta}\ldots m \text{ times} \\ &= \epsilon^{mr\theta}\phi(D+mr)\phi(D+mr-r)\ldots\ldots\phi(D+r), \end{aligned}$$

we have

$$\begin{aligned} u_1 = A\{P + \epsilon^{r\theta}\phi(D+r)P + \epsilon^{2r\theta}\phi(D+2r)\phi(D+r)P \\ + \epsilon^{3r\theta}\phi(D+3r)\phi(D+2r)\phi(D+r)P + \ldots\}, \end{aligned}$$

in which it only remains to perform the operations indicated by $\phi(D+r)$, by $\phi(D+2r)\phi(D+r)$, ... on the function P.

Let us in the first place suppose the symbolic function $f_0(D)$ to be of the form $(D-a)(D-b)\ldots$; then

$$\{f_0(D)\}^{-1}0 = A\epsilon^{a\theta} + B\epsilon^{b\theta} + \ldots.$$

Here $P = \epsilon^{a\theta}$. Hence substituting in the above expression for u, and observing that $f(D)\epsilon^{n\theta} = f(n)\epsilon^{n\theta}$, we find

$$u_1 = A\epsilon^{a\theta}\{1 + \phi(a+r)\epsilon^{r\theta} + \phi(a+2r)\phi(a+r)\epsilon^{2r\theta} + \ldots\},$$

or, since $\epsilon^{\theta} = x$,

$$u_1 = Ax^a\{1 + \phi(a+r)x^r + \phi(a+2r)\phi(a+r)x^{2r} + \ldots\};$$

and

$$u = Ax^a \{1 + \phi(a+r)\, x^r + \phi(a+2r)\, \phi(a+r)\, x^{2r} + \ldots\}$$
$$+ Bx^b \{1 + \phi(b+r)\, x^r + \phi(b+2r)\, \phi(b+r)\, x^{2r} + \ldots\}$$
$$+ \ldots,$$

the solution sought.

Consider now the general equation

$$f_0(D)\, u + f_1(D)\, \epsilon^{\theta} u + \ldots + f_n(D)\, \epsilon^{n\theta} u = 0.$$

Here we have, representing $\dfrac{f_m(D)}{f_0(D)}$ by $\phi_m(D)$,

$$\{1 + \phi_1(D)\, \epsilon^{\theta} + \ldots. + \phi_n(D)\, \epsilon^{n\theta}\}\, u = \{f_0(D)\}^{-1} 0;$$

therefore

$$u = \{1 + \phi_1(D)\, \epsilon^{\theta} + \ldots\ldots\ \phi_n(D)\, \epsilon^{n\theta}\}^{-1} \{f_0(D)\}^{-1} 0.$$

Here we have first to determine $\{f_0(D)\}^{-1} 0$, then to determine the effect of the operation represented by

$$\{1 + \phi_1(D)\, \epsilon^{\theta} + \ldots\ldots\ \phi_n(D)\, \epsilon^{n\theta}\}^{-1}$$

upon this.

Now $\{f_0(D)\}^{-1} 0$ is given by the solution of a linear differential equation with constant coefficients, and will therefore be of the form

$$AP + BQ + CR + \ldots,$$

A, B, C, ... being arbitrary constants, and P, Q, R, ... functions of θ.

Again, since

$$\{1 + \phi_1(D)\, \epsilon^{\theta} + \ldots\ldots + \phi_n(D)\, \epsilon^{n\theta}\}^{-1}$$
$$= \frac{1}{1 + \phi_1(D)\epsilon^{\theta} + \ldots\ldots + \phi_n(D)\, \epsilon^{n\theta}},$$

it may be shewn by a process of actual symbolical division, attending to the laws of combination of symbols, that the expression may be expanded in the form

$$F_0(D) + F_1(D)\,\epsilon^\theta + F_2(D)\,\epsilon^{2\theta} + \ldots.$$

To determine the functions $F_0(D)$, $F_1(D)$,...... we may proceed as follows. From the equation

$$\{1 + \phi_1(D)\,\epsilon^\theta + \ldots\ldots\ldots + \phi_n(D)\,\epsilon^{n\theta}\}^{-1}$$
$$= F_0(D) + F_1(D)\,\epsilon^\theta + F_2(D)\,\epsilon^{2\theta} + \ldots$$

we have

$$1 = \{1 + \phi_1(D)\,\epsilon^\theta + \ldots + \phi_n(D)\,\epsilon^{n\theta}\}\{F_0(D) + F_1(D)\,\epsilon^\theta + F_2(D)\,\epsilon^{2\theta} + \ldots\}$$
$$= F_0(D) + \{F_1(D) + \phi_1(D)\,F_0(D-1)\}\,\epsilon^\theta + \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Hence $$F_0(D) = 1,$$

$$F_1(D) + \phi_1(D)\,F_0(D-1) = 0;$$

therefore $$F_1(D) = -\,\phi_1(D)\,F_0(D-1),$$

and so on. Hence $F_0(D)$, $F_1(D)$,...... are determined in succession. The general law is as follows: the coefficient of $\epsilon^{m\theta}$ in the second member of (1), when m is greater than 1, is

$$F_m(D) + \phi_1(D)\,F_{m-1}(D-1) + \phi_2(D)\,F_{m-2}(D-2) + \ldots \quad (2),$$

whence

$$F_m(D) = -\,\phi_1(D)\,F_{m-1}(D-1) - \phi_2(D)\,F_{m-2}(D-2) - \ldots.$$

By this formula the successive values of $F_m(D)$ can be deduced from those of $F_{m-1}(D)$, $F_{m-2}(D)$, ...

Combining the above results we obtain thus for u the expression

$$u = \{1 + F_1(D)\,\epsilon^\theta + F_2(D)\,\epsilon^{2\theta} + \ldots.\}\{AP + BQ + \ldots\}$$
$$= A\{P + F_1(D)\,\epsilon^\theta P + F_2(D)\,\epsilon^{2\theta} P + \ldots.\}$$
$$+ B\{Q + F_1(D)\,\epsilon^\theta Q + F_2(D)\,\epsilon^{2\theta} Q + \ldots.\}$$
$$+ \ldots.$$

Let us in applying this expression first suppose that the factors of $f_0(D)$ are real and unequal, so that $f_0(D)$ is of the form $(D-a)(D-b)(D-c)$.... Further, let us suppose that no two of the quantities $a, b, c, \ldots$ differ by an integer.

Then $$\{f_0(D)\}^{-1}0 = A\epsilon^{a\theta} + B\epsilon^{b\theta} + \ldots,$$
whence we may assume
$$P = \epsilon^{a\theta}, \quad Q = \epsilon^{b\theta}, \ldots$$

Thus the expression for u becomes
$$A\{\epsilon^{a\theta} + F_1(D)\,\epsilon^{(a+1)\theta} + F_2(D)\,\epsilon^{(a+2)\theta} + \ldots\}$$
$$+ B\{\epsilon^{b\theta} + F_1(D)\,\epsilon^{(b+1)\theta} + F_2(D)\,\epsilon^{(b+2)\theta} + \ldots\}$$
$$+ \ldots;$$
or, since $$F(D)\,\epsilon^{m\theta} = F(m)\,\epsilon^{m\theta},$$
$$u = A\{\epsilon^{a\theta} + F_1(a+1)\,\epsilon^{(a+1)\theta} + F_2(a+2)\,\epsilon^{(a+2)\theta} + \ldots\}$$
$$+ B\{\epsilon^{b\theta} + F_1(b+1)\,\epsilon^{(b+1)\theta} + F_2(b+2)\,\epsilon^{(b+2)\theta} + \ldots\}$$
$$+ \ldots$$

Hence, replacing ϵ^θ by x,
$$u = A\{x^a + F_1(a+1)\,x^{a+1} + F_2(a+2)\,x^{a+2} + \ldots\}$$
$$+ B\{x^b + F_1(b+1)\,x^{b+1} + F_2(b+2)\,x^{b+2} + \ldots\}$$
$$+ \ldots$$

In (2) replace in like manner D by $a+i$ and we have, putting i for m,
$$F_i(a+i) + \phi_1(a+i)\,F_{i-1}(a+i-1)$$
$$+ \phi_2(a+i)\,F_{i-2}(a+i-2) + \ldots\ldots = 0,$$
or, if $F_i(a+i)$ be represented by u_{a+i},
$$u_{a+i} + \phi_1(a+i)\,u_{a+i-1} + \phi_2(a+i)\,u_{a+i-2} + \ldots\ldots = 0.$$

Put m for $a+i$, thus
$$u_m + \phi_1(m)\,u_{m-1} + \phi_2(m)\,u_{m-2} + \ldots\ldots = 0.$$

This agrees with the law established in [there is no reference in the manuscript, but the law intended appears to be that given in Chap. XVII. Art. 9.]

Secondly, suppose that r of the factors of $f_0(D)$ are equal and of the form $D-a$.

Then $\{f_0(D)\}^{-1}0$ contains a term of the form

$$\epsilon^{a\theta}(c_0+c_1\theta+c_2\theta^2+\ldots\ldots+c_{r-1}\theta^{r-1}).$$

Hence the corresponding portion of u is of the form

$$\left\{1+F_1(D)\,\epsilon^{\theta}+F_2(D)\epsilon^{2\theta}+\ldots.\right\}\epsilon^{a\theta}(c_0+c_1\theta+\ldots\ldots+c_{r-1}\theta^{r-1})$$

$$=\left\{\epsilon^{a\theta}+\epsilon^{(a+1)\theta}F_1(D+a+1)+\epsilon^{(a+2)\theta}F_2(D+a+2)+\ldots\right\}v\ldots(3)$$

where v stands for $c_0+c_1\theta+c_2\theta^2+\ldots.+c_{r-1}\theta^{r-1}$.

Now $\quad F_i(D+a+i)\,v$

$$=\left\{F_i(a+1)+F_i'(a+1)D+F_i''(a+1)\frac{D^2}{1.2}+\ldots.\right\}v,$$

which on performing the differentiations becomes a polynomial of the form

$$A_0+A_1\theta+\ldots\ldots+A_{r-1}\theta^{r-1}.$$

We see thus that (3) will assume the form of a series of terms $\epsilon^{a\theta}$, $\epsilon^{(a+1)\theta}$, ... each multiplied by a polynomial of the $(r-1)^{\text{th}}$ degree in θ. Or arranging the terms otherwise it will consist of a series of terms of the form

$$B_0+B_1\theta+\ldots\ldots+B_{r-1}\theta^{r-1},$$

in which B_0, B_1, ... B_{r-1} are series involving $\epsilon^{a\theta}$, $\epsilon^{(a+1)\theta}$, $\epsilon^{(a+2)\theta}$,..... Or lastly, changing ϵ^{θ} to x, the portion of u in question is of the form

$$B_0+B_1(\log x)+\ldots\ldots+B_{r-1}(\log x)^{r-1},$$

B_0, B_1, ... B_{-1} being polynomials in each of which the lowest power of x is x^a, and the successive powers increase by unity.

This establishes the assumption in [there is no reference in the manuscript; probably Chap. XVII. Art. 10 is to be supplied.]

Thirdly, let $f_0(D)$ contain r factors $D-a_1$, $D-a_2, \ldots D-a_r$ in which $a_1, a_2, \ldots a_r$ differ from each other by integers, together with other factors.

The portion of u corresponding to the factor $D-a_t$ will be

$$\{1+F_1(D)\,\epsilon^{\theta}+F_2(D)\,\epsilon^{2\theta}+\ldots.\}A\epsilon^{a_t\theta},$$

in which

$$F_m(D)\,\epsilon^{m\theta}=-\Big\{\phi_1(D)\,\epsilon^{\theta}F_{m-1}(D)\,\epsilon^{(m-1)\theta}$$
$$+\phi_2(D)\,\epsilon^{2\theta}F_{m-2}(D)\,\epsilon^{(m-2)\theta}+\ldots.\Big\}.$$

Thus $F_m(D)\,\epsilon^{m\theta}$ consists of terms of the form

$$\phi_i(D)\,\epsilon^{i\theta}F_{m-i}(D)\,\epsilon^{(m-i)\theta},$$

i being one of the numbers 1, 2, ... n. Hence $F_{m-i}(D)\,\epsilon^{(m-i)\theta}$ will consist of terms of the form

$$\phi_j(D)\,\epsilon^{j\theta}F_{m-i-j}(D)\,\epsilon^{(m-i-j)\theta},$$

j being one of the numbers 1, 2, ... n. Continuing this until $i+j+k+\ldots=m$, we see that $F_m(D)\,\epsilon^{m\theta}$ will ultimately consist of terms of the form

$$\phi_i(D)\,\epsilon^{i\theta}\phi_j(D)\,\epsilon^{j\theta}\phi_k(D)\,\epsilon^{k\theta}\ldots,$$

$i, j, k, \ldots$ receiving arbitrarily any of the values 1, 2, ... n, and $i+j+k+\ldots$ being equal to m.

Thus the portion of u derived from $A\epsilon^{a_t\theta}$ will consist of all possible terms of the form

$$A\phi_i(D)\,\epsilon^{i\theta}\phi_j(D)\,\epsilon^{j\theta}\phi_k(D)\,\epsilon^{k\theta}\ldots.(\epsilon^{a_t\theta})$$

$$=A\phi_i(D)\,\phi_j(D-i)\,\phi_k(D-i-j)\ldots.\epsilon^{(m+a_t)\theta}$$

$$=\frac{Af_i(D)\,f_j(D-i)\,f_k(D-i-j)\ldots}{f_0(D)f_0(D-i)f_0(D-i-j)\ldots}\,\epsilon^{(m+a_t)\theta}.$$

Let $i=\alpha$, $i+j=\beta$, ... $i+j+k+$ ..... excluding the last term $=\mu$; and let the symbolical numerator which involves only direct functions be represented by $f(D)$, and we have

$$\frac{Af(D)}{f_0(D)f_0(D-\alpha)f_0(D-\beta)\ldots\ldots f_0(D-\mu)}\,\epsilon^{(m+a_t)\theta},$$

in which α, β, ... μ, m are integers ascending by differences not exceeding n.

[A few lines of the manuscript here are obscure, and I venture to express in other words the idea which seems to be involved.

Let $D-a_s$ denote one factor of $f_0(D)$, then the corresponding factors in the denominator of

$$\frac{Af(D)\,\epsilon^{(m+a_t)\theta}}{f_0(D)f_0(D-\alpha)f_0(D-\beta)\ldots f_0(D-\mu)}\ldots\ldots\ldots(4),$$

are $\quad (D-a_s)(D-a_s-\alpha)\ \ldots\ldots\ (D-a_s-\mu)\ \ldots\ldots\ldots(5).$

Now if a_s is not greater than a_t, then $a_s+\mu$ is less than a_t+m; hence no factor in the expression (5) can be identical with $D-m-a_t$. But if a_s is greater than a_t, then *one* factor in the expression (5) may be identical with $D-m-a_t$.

Hence it follows that the denominator of the expression (4) *may* contain $D-m-a_t$ to the power $r-1$, but not to a higher power.]

And, since

$$(D-m-a_t)^{-(r-1)}\,\epsilon^{(m+a_t)\theta}$$

$$=\epsilon^{(m+a_t)\theta}\left\{c_0+c_1\theta+\ldots+c_{r-2}\theta^{r-2}+\frac{\theta^{r-1}}{\lfloor r-1}\right\},$$

we see that u will contain r sets of terms together of the form

$$A+B(\log x)+C(\log x)^2+\ldots\ldots+K(\log x)^{r-1},$$

A, B, C.... being polynomials in x.

This establishes the rule in [there is no reference in the manuscript; probably Chap. XVII. Art. 10 is to be supplied.]

[There is no hint in the manuscript as to the position which Article 2 was intended to occupy; and the reasoning does not seem fully developed.]

2. PROP. The solution of the equation

$$f_0(D)\,u + f_1(D)\,\epsilon^{\theta}u + \ldots\ldots + f_n(D)\,\epsilon^{n\theta}u = 0$$

being expressed in the form

$$\{1 + F_1(D)\,\epsilon^{\theta} + F_2(D)\,\epsilon^{2\theta} + \ldots. +\}\,\{f_0(D)\}^{-1}0,$$

it is not necessary to introduce new constants in interpreting $F_1(D)$, ...; it suffices to interpret particularly if only uniformly and consistently.

For let

$$\{f_0(D)\}^{-1}0 = AP + BQ + \ldots\ldots;$$

and in interpreting

$$F_m(D)\,\epsilon^{m\theta}\,(AP + BQ + \ldots\ldots)$$

let a new constant be introduced which was not in the interpretation of

$$F_{m-1}(D)\,\epsilon^{(m-1)\theta}\,(AP + BQ + \ldots\ldots).$$

Now $$F_m(D)\,\epsilon_m^{\theta} + \phi_1(D)\,\epsilon^{\theta}F_{m-1}(D)\,\epsilon^{(m-1)\theta} + \phi_2(D)\,\epsilon^{2\theta}F_{m-2}(D)\,\epsilon^{(m-2)\theta} + \ldots\ldots = 0,$$

therefore

$$F_m(D)\,\epsilon^{m\theta} = -\,\phi_1(D)\,\epsilon^{\theta}F_{m-1}(D)\,\epsilon^{(m-1)\theta} - \ldots.$$

$$= -\,\{f_0(D)\}^{-1}\,\{f_1(D)\,\epsilon^{\theta}F_{m-1}(D)\,\epsilon^{(m-1)\theta} + \ldots\},$$

hence the new constant comes from $\{f_0(D)\}^{-1}0$, and the term containing it must be $A'P$, or $B'Q$,...., where A', B'.... are constants. Suppose it $A'P$;

then as derived from this,

$$F_{m+1}(D)\,\epsilon^{(m+1)\theta}\{f_0(D)\}^{-1}\,0 = -\,\phi_1(D)\,\epsilon^{\theta}A'P,$$

$$F_{m+2}(D)\,\epsilon^{(m+2)\theta}\,\{f_0(D)\}^{-1}\,0 = -\,\phi_1(D)\,\epsilon^{\theta}F_{m+1}(D)\epsilon^{(m+1)\theta}A'P - \phi_2(D)\,\epsilon^{2\theta}F_m(D)\,\epsilon^{m\theta}A'P.$$

Thus $$F_{(m+i)}(D)\epsilon^{(m+i)\theta} = -\phi_1(D)\,\epsilon^{\theta} F_{m+i-1}\epsilon^{(m+i-1)\theta}$$
$$-\phi_2(D)\,\epsilon^{\theta}\,F_{m+i-2}\epsilon^{(m+i-2)\theta}$$
$$-\ldots.$$

The law of derivation is exactly the same as in the derivation of $F_i(D)\,\epsilon^{i\theta}$ from $F_{i-1}(D)\,\epsilon^{(i-1)\theta}$,......

[Art. 3 seems intended for a reconstruction on an extended scale of part of Chapter XVII. Art. 3.]

3. We proceed to consider more fully the theory of the binomial equation

$$u + \phi(D)\,\epsilon^{r\theta} u = U.$$

Now the possibility of solving the equation depends upon the nature of the symbolic function $\phi(D)$. It is perhaps the most general account of the present state of the theory to say that there exist certain *primary* forms of this function which render the equation solvable, and that to each of these primary forms an infinite number of the forms are reducible by general theorems of transformation. As these theorems admit of a statement which is independent of the form of the function $\phi(D)$, we shall establish them first.

PROP. II. *The function $\phi(D)$ in the equation*

$$u + \phi(D)\,\epsilon^{r\theta} u = U$$

can without otherwise changing the first member of that equation be 1st affected with any constant factor, or 2ndly converted into $\phi(D+a)$, or 3rdly converted into $\{\phi(-D)\}^{-1}$.

First. Let $U = f(\epsilon^{\theta})$, and in the equation

$$u + \phi(D)\,\epsilon^{r\theta} u = f(\epsilon^{\theta}),$$

let $\epsilon^{\theta} = a^{\frac{1}{r}}\epsilon^{\theta'}$. Then $\dfrac{d}{d\theta} = \dfrac{d}{d\theta'}$, and the equation becomes

$$u + a\phi(D)\,\epsilon^{r\theta'} u = f(a^{\frac{1}{r}}\epsilon^{\theta'}),$$

in which $D = \frac{d}{d\theta'}$. Thus $\phi(D)$ has been affected by a constant factor a.

Secondly. In the same system let $u = \epsilon^{a\theta} v$. Then

$$\epsilon^{a\theta} v + \phi(D)\, \epsilon^{(a+r)\theta} v = f(\epsilon^{\theta}),$$

or

$$\epsilon^{a\theta} v + \epsilon^{a\theta} \phi(D + a)\, \epsilon^{r\theta} v = f(\epsilon^{\theta}),$$

therefore

$$v + \phi(D + a)\, \epsilon^{r\theta} v = \epsilon^{-a\theta} f(\epsilon^{\theta}).$$

Here $\phi(D)$ has been changed into $\phi(D + a)$.

The result of this transformation may be conveniently expressed by the following theorem.

The equation

$$u + \phi(D)\, \epsilon^{r\theta} u = U$$

will be converted into

$$v + \phi(D + a)\, \epsilon^{r\theta} v = V$$

by the relations

$$u = \epsilon^{a\theta} v, \quad U = \epsilon^{a\theta} V.$$

Thirdly. In the same equation let $\theta = -\theta'$; then

$$\frac{d}{d\theta} = -\frac{d}{d\theta'},$$

and we have

$$u + \phi(-D)\, \epsilon^{-r\theta'} u = f(\epsilon^{-\theta'}),$$

in which $D = \frac{d}{d\theta'}$. Hence

$$u + \epsilon^{-r\theta'} \phi\{-(D - r)\}\, u = f(\epsilon^{-\theta'});$$

therefore

$$\epsilon^{r\theta'} u + \phi(r - D)\, u = \epsilon^{r\theta'} f(\epsilon^{-\theta'}),$$

whence

$$u + \{\phi(r - D)\}^{-1} \epsilon^{r\theta'} u = \{\phi(r - D)\}^{-1} \epsilon^{r\theta'} f(\epsilon^{-\theta'}).$$

In this equation let $u = \epsilon^{r\theta'} v$. Then by the last theorem,

$$v + \{\phi(-D)\}^{-1} \epsilon^{r\theta'} v = \{\phi(-D)\}^{-1} f(\epsilon^{-\theta'}).$$

Thus

$$u + \phi(D)\,\epsilon^{r\theta} u = f(\epsilon^{\theta})$$

is converted into

$$v + \{\phi(-D)\}^{-1} \epsilon^{r\theta'} v = \{\phi(-D)\}^{-1} f(\epsilon^{-\theta'}),$$

in which $D = \dfrac{d}{d\theta'}$, *by assuming*

$$\theta = -\theta', \quad u = \epsilon^{r\theta'} v.$$

The above transformations leave the index r in the first member unchanged. If however we assume $\theta = \dfrac{\theta'}{a}$, whence $\dfrac{d}{d\theta} = a\dfrac{d}{d\theta'}$, we should have

$$u + \phi(aD)\epsilon^{\frac{r\theta'}{a}} = f\left(\epsilon^{\frac{\theta'}{a}}\right).$$

By combining this with the previous results we see that it is possible to convert $\phi(D)$ into $\phi(aD+b)$, and into $\{\phi(aD+b)\}^{-1}$.

But the most important transformation of the function $\phi(D)$ is that which is established in the following proposition.

[The proposition referred to is Prop. III. of Chap. XVII. Art. 3.]

[Article 4 was intended to follow the words "or subsequently in the derivation of u" in Chap. XVII. Art. 4.]

4. It becomes therefore important to establish rules for the treatment of the constants which in these different ways arise.

Now the entire process of solution consists of three stages, namely:

1st, the determination of V by the equation

$$V = P_r \frac{\psi(D)}{\phi(D)} U.$$

2ndly, the solution of the transformed equation

$$v + \psi(D)\,\epsilon^{r\theta} v = V,$$

3rdly, the determination of u by the relation

$$u = P_r \frac{\phi(D)}{\psi(D)} v.$$

Let us consider these separately, supposing $\phi(D)$ to contain a single factor $\dfrac{D+a}{D+b}$ which is made to disappear in the generation of $\psi(D)$, so that a and b differ by a multiple of r. Thus the given equation is of the form

$$u - \frac{D+a}{D+b}\psi(D)\,\epsilon^{r\theta} u = U \text{.............} (6).$$

The transformed equation is of the form

$$v - \psi(D)\,\epsilon^{r\theta} v = V,$$

in which $\quad u = P_r \dfrac{D+a}{D+b} v, \quad V = P_r \dfrac{D+b}{D+a} U.$

First, suppose $a - b = nr$, where n is positive.

Thus

$$u = (D+a)(D+a-r)\ldots(D+a-nr+r)\,v,$$

$$V = \left\{(D+a)(D+a-r)\ldots(D+a-nr+r)\right\}^{-1} U.$$

Hence

$$u = (D+a)\ldots(D+a-nr+r)\{1-\psi(D)\epsilon^{\theta}\}^{-1}V$$

$$= (D+a)\ldots(D+a-nr+r)\{1-\psi(D)\epsilon^{r\theta}\}^{-1}\Big\{U_1 + C_1\epsilon^{-a\theta} + C_2\epsilon^{-(a-r)\theta} + \ldots\ldots + C_n\epsilon^{-(a-nr+r)\theta}\Big\} \ldots.(7),$$

where U_1 is a particular value of

$$\Big\{(D+a)\ldots.(D+a-nr+r)\Big\}^{-1}U.$$

The part containing the constants will consist of terms of the form

$$(D+a)\ldots(D+a-nr+r)\{1-\psi(D)\epsilon^{r\theta}\}^{-1}C\epsilon^{-(a-ir)\theta}$$

$$= (D+a)(D+a-r)\ldots(D+a-nr+r)\Big\{1+\psi(D)\epsilon^{r\theta} + \psi(D)\epsilon^{r\theta}\psi(D)\epsilon^{r\theta} + \ldots\Big\}C\epsilon^{-(a-ir)\theta}$$

$$= C(D+a)(D+a-r)\ldots(D+a-nr+r)\Big\{\epsilon^{-(a-ir)\theta} + \psi(D)\epsilon^{-(a-ir-r)\theta} + \psi(D)\psi(D-r)\epsilon^{-(a-ir-2r)\theta} + \ldots\Big\}.$$

Now all these terms vanish up to the one containing $\epsilon^{-(a-nr)\theta}$; therefore we have to perform the operation

$C(D+a)(D+a-r)\ldots(D+a-nr+r)$ on

$$\Big\{\psi(D)\psi(D-r)\ldots\psi(D-jr)\epsilon^{-(a-nr)\theta} + \psi(D)\psi(D-r)\ldots\psi(D-jr-r)\epsilon^{-(a-nr-r)\theta} + \ldots.\Big\},$$

where $j = n-i-1$; that is, we have to perform the operation

$C(D+a)(D+a-r)\ldots(D+a-nr+r)$ on

$$\Big\{\psi(D)\psi(D-r)\ldots\psi(D-jr)\epsilon^{-(a-nr)\theta} + \psi(D)\epsilon^{\theta}\psi(D)\ldots\psi(D-jr)\epsilon^{-(a-nr)\theta} + \ldots.\Big\}.$$

Now $\psi(D)\,\psi(D-r)\,..\,\psi(D-jr)\,\epsilon^{-(a-nr)\theta}$

$$=\psi(nr-a)\,\psi(nr-r-a)\,\dots\,\psi(nr-jr-a)\,\epsilon^{-(a-nr)\theta}$$
$$=B\epsilon^{-(a-nr)\theta};$$

therefore we obtain

$$BC(D+a)\dots(D+a-nr+r)\left\{\epsilon^{-(a-nr)\theta}+\psi(D)\,\epsilon^{\theta}\epsilon^{-(a-nr)\theta}+\dots\right\}$$
$$=BC(D+a)\dots(D+a-nr+r)\left\{1+\psi(D)\,\epsilon^{\theta}\right.$$
$$\left.+\psi(D)\,\epsilon^{\theta}\psi(D)\,\epsilon^{\theta}+\dots\right\}\epsilon^{-(a-nr)\theta}.$$

Thus this expression is the same in form for all values of i. Therefore all the terms containing an arbitrary constant in (7) are equivalent to only one term.

Secondly, suppose $a-b=-nr$.

Then $u=\left\{(D+b)\,(D+b-r)\dots(D+b-nr+r)\right\}^{-1}v$,

$$V=(D+b)\,(D+b-r)\,\dots\,(D+b-nr+r)\,U.$$

Here there are no constants in V. But u contains n arbitrary constants not in v, and as there is no subsequent process in the method for destroying these or reducing them to mutual dependence, it is necessary that the relations connecting them should be sought by comparing the solution with that given by the method of development in series.

NOTE. It would be better to reduce (6) to the form

$$u-\frac{D}{D-a}\psi(D)\,\epsilon^{r\theta}u=U$$

before the demonstration.

[Article 5 was intended to follow Chap. XVII. Art. 7.

There is a memoir by Professor Boole on the subject of this Article, entitled *On the Differential Equations which determine the form of the Roots of Algebraic Equations.* The memoir occupies pages 733—755 of the *Philosophical Transactions* for 1864.]

5. If we agree to regard as *primary* those forms of binomial equations which are integrable but not through any reduction effected by the Propositions of Art. 3, and to which equations through the application of those propositions other equations are reducible and so made integrable, it becomes very important to enquire what these primary integrable forms are. It does not appear at present possible to give a general answer to this question, but so far as is known, such forms if belonging to differential equations of a degree higher than the first stand in a remarkable connexion with the theory of algebraical equations. By the study of this theory Mr Harley was led to the conclusion that y defined as an implicit function of x by the algebraical equation

$$y^n - ny + (n-1)x = 0 \qquad (8),$$

n being greater than 2, satisfies the binomial differential equation

$$y - \frac{\left(D - \frac{2n-1}{n}\right)\left(D - \frac{3n-2}{n}\right)\cdots\left(D - \frac{n^2-n+1}{n}\right)}{D(D-1) \quad \cdots \quad (D-n+2)} \epsilon^{(n-1)\theta} y = 0,$$

in which $\epsilon^\theta = x$. In this expression the factors of the numerator are equidifferent, as of the denominator, their common difference being $\frac{n-1}{n}$, but the equation is not resolvable by Propositions II. and III. into forms, the integrability which had before been recognised.

The above result first reached by induction was confirmed by Mr Cayley by the aid of Lagrange's theorem.

To the form (8) all algebraic equations of the third, fourth, and fifth degrees are known to be reducible.

Mr Harley has subsequently found that y considered as a function of x defined implicitly by the equation

$$y^n - ny^{n-1} + (n-1)x = 0$$

satisfies the symbolical differential equation

$$n^{n-1}[(n-1)D]^{n-1}y - (n-1)(nD-n-1)[nD-2]^{n-2}e^\theta y = [n-1]^{n-1}e^\theta$$

the factorial notation according to which

$$[m]^n = m(m-1)(m-2)\dots(m-n+1)$$

being here adopted.

These results are implicitly involved in a more general theorem which I shall now demonstrate.

THEOREM. *If $y_1, y_2 \dots\dots y_n$ are the n roots of the algebraic equation*

$$y^n - ay^{n-1} + 1 = 0,$$

and if the m^{th} power of any one of these roots be represented by u, and $\log a$ by θ, then u as a function of θ satisfies the differential equation

$$u - \frac{\left[\frac{n-1}{n} D + \frac{m}{n} - 1\right]^{n-1} \left(\frac{D}{n} - \frac{m}{n} - 1\right)}{[D]^n} e^{n\theta} u = 0.$$

And the complete integral of the above differential equation will be

$$u = C_1 y_1^m + C_2 y_2^m \dots\dots + C_n y_n^m.$$

Let $y^n = z$, then the given equation may be expressed in the form

$$z = b + az^{\frac{n-1}{n}},$$

in which $b = -1$. Hence, by Lagrange's theorem,

$$u = z^{\frac{m}{n}} = b^{\frac{m}{n}} + ab^{\frac{n-1}{n}} \frac{d}{db} b^{\frac{m}{n}}$$

$$+ \frac{1}{1.2} \frac{d}{db} \left\{ \left(ab^{\frac{n-1}{n}}\right)^2 \frac{d}{db} b^{\frac{m}{n}} \right\} + \&c.,$$

the general term being

$$\frac{1}{1.2\dots r} \left(\frac{d}{db}\right)^{r-1} \left\{ \left(ab^{\frac{n-1}{n}}\right)^r \frac{d}{db} b^{\frac{m}{n}} \right\},$$

which on effecting the differentiations and adopting the factorial notation becomes

$$\frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1} b^{\frac{m-r}{n}}}{n\,[r]^r}\,a^r,$$

and this expression will be found to represent the first term as well as the others of Lagrange's expansion provided that we interpret the form

$$[p]^0 \text{ by } 1, \text{ and } [p]^{-1} \text{ by } \frac{1}{1+p}.$$

Further, the above general development includes the n particular developments of u or $y^{\frac{m}{n}}$ arising from the giving to $b^{\frac{1}{n}}$ its n particular algebraic values. In this way it represents the m^{th} power of each of the n roots $y_1, y_2, \ldots y_n$ in succession.

Now representing the above general term by $u_r a^r$, we shall have

$$u_r = \frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1} b^{\frac{m-r}{n}}}{n\,[r]^r},$$

$$u_{r-n} = \frac{m\left[\frac{m+(n-1)r}{n}-n\right]^{r-n-1} b^{\frac{m-r}{n}+1}}{n\,[r-n]^{r-n}}.$$

Therefore, after reduction and replacing b by -1,

$$\frac{u_r}{u_{r-n}} = \frac{\left[\frac{m+(n-1)r}{n}-1\right]^{n-1}\left(\frac{r}{n}-\frac{m}{n}-1\right)}{[r]^n} \quad \ldots\ldots (9).$$

It follows therefore that the complete series of which the general term is $u_r a^r$ will if represented by u satisfy the differential equation

$$u - \frac{\left[\frac{n-1}{n} D + \frac{m}{n} - 1\right]^{n-1} \left(\frac{D}{n} - \frac{m}{n} - 1\right)}{[D]^n} \epsilon^{n\theta} u = 0 \ldots (\mathrm{I}).$$

If we integrate the equation in a series (Chap. XVII. Art. 9), the initial terms of the value of u will be

$$C_0 + C_1 a + C_2 a^2 \ldots\ldots + C_{n-1} a^{n-1},$$

the succeeding terms being formed from these by the law (9). Hence, if the arbitrary constants $C_0 C_1 \ldots\ldots C_{n-1}$ be so determined as to make the above initial terms agree with the first n terms of the Lagrangean expansion in any of its particular forms, the succeeding terms will also agree, and the Lagrangean expansion will thus become a particular integral of the equation (I). The aggregate of such particular integrals, each affected by an arbitrary constant, will therefore also be an integral of the differential equation, and will, in fact, constitute its general integral, subject to exception only in the case in which for a particular value of m the integrals $y_1^m, y_2^m, \ldots\ldots y_n^m$ cease to be independent.

For instance, if $m = -1$, and we reduce the equation to the form

$$(y^{-1})^n - ay^{-1} + 1 = 0,$$

it is seen that except when $n = 2$, we have

$$y_1^{-1} + y_2^{-1} \ldots\ldots + y_n^{-1} = 0.$$

Here then the solution

$$u = C_1 y_1^m + C_2 y_2^m \ldots\ldots + C_n y_n^m \ldots\ldots\ldots\ldots\ldots (10)$$

ceases to be general for it becomes

$$u = (C_1 - C_n)\, y_1^{-1} + (C_2 - C_n)\, y_2^{-1} \ldots + (C_{n-1} - C_n)\, y^{-1}_{n-1},$$

and virtually involves but $n-1$ arbitrary constants.

If, however, we give to the integral the form

$$u = C_1 y_1^m + C_2 y_2^m \ldots + C_{n-1} y^m_{n-1} + C_n \frac{y_1^m + y_2^m \ldots + y_n^m}{m+1},$$

the last term of which becomes a vanishing fraction when $m = -1$, we find for the general value of u in this case

$$u = C_1 y_1^{-1} + C_2 y_2^{-1} \ldots\ldots + C_{n-1} y^{-1}_{n-1}$$
$$+ C_n (y_1^{-1} \log y_1 + y_2^{-1} \log y_2 \ldots\ldots + y_n^{-1} \log y_n),$$

and in this way we may proceed in failing cases generally.

Lastly, it may be observed that in certain cases the differential equation (I) admits of reduction to an order lower by unity than its own. And in particular this happens in the failing cases above noticed. Thus, if in (I) we make $m = -1$ the equation will be expressible in the form

$$D(D-1) \ldots (D-n+1)\, u$$
$$- \frac{1}{n}\left[\frac{n-1}{n} D - \frac{1}{n} - 1\right]^{n-1} (D-n+1)\, \epsilon^{n\theta} u = 0,$$

whence, operating on both members with $(D-n+1)^{-1}$, we have

$$[D]^{n-1} u - \frac{1}{n}\left[\frac{n-1}{n} D - \frac{1}{n} - 1\right]^{n-1} \epsilon^{n\theta} u = C\epsilon^{(n-1)\theta}.$$

The general integral of this equation will be expressed by (10) provided that a proper relation be established between C and the constants $C_1, C_2, \ldots\ldots C_n$. If we choose to determine C so as to give to the integral the particular form y^{-1}, we shall find on substituting for u its Lagrangean development making $m = -1$, $b = -1$, and calculating the coefficient of a^{n-1} or $\epsilon^{(n-1)\theta}$ in the first member of the differential,

$$C = \frac{[n-3]^{n-2}}{n}.$$

Hence, if n be greater than 2, we have $C = 0$. It follows therefore that if n be greater than 2, the equation

$$[D]^{n-1} u - \frac{1}{n}\left[\frac{n-1}{n} D - \frac{1}{n} - 1\right]^{n-1} \epsilon^{n\theta} u = 0 \ldots\ldots (\text{II}),$$

in which $\epsilon^\theta = a$ has for its general integral

$$u = C_1 y_1^{-1} + C_2 y_2^{-1} \dots + C_{n-1} y^{-1}_{n-1},$$

$y_1, y_2, \dots y_{n-1}$ being any $n-1$ roots of the equation

$$y^n - ay^{n-1} + 1 = 0.$$

It may be useful to notice the forms which the above results assume when θ is changed into $-\theta$, and therefore D into $-D$; see Art. 3.

It will be found that (I) becomes

$$u - \frac{[D-1]^n}{\left[\frac{n-1}{n} D - \frac{m}{n}\right]^{n-1} \left(\frac{D}{n} + \frac{m}{n}\right)} \epsilon^{n\theta} u = 0 \dots\dots \text{(III)},$$

of which the integral is therefore

$$u = C_1 y_1^m + C_2 y_2^m \dots\dots + C_n y_n^m,$$

$y_1, y_2, \dots y_n$ being the roots of the equation

$$y^n - \frac{1}{a} y^{n-1} + 1 = 0 \dots\dots\dots\dots\dots\dots (11);$$

and $\log a$ being denoted by θ;

while as the equivalent of (II) we have

$$u - n \frac{[D-2]^{n-1}}{\left[\frac{n-1}{n} D + \frac{1}{n}\right]^{n-1}} \epsilon^{n\theta} u = 0 \dots\dots\dots\dots \text{(IV)},$$

of which, supposing n greater than 2, the integral is

$$u = C_1 y_1^{-1} + C_2 y_2^{-1} \dots + C_{n-1} y^{-1}_{n-1},$$

$y_1, y_2, \dots y_{n-1}$ being any $n-1$ roots of the same algebraic equation.

Mr Harley's results may readily be deduced from the above. Thus it will be found that the equation (11) reduces to

$$t^n - nt + (n-1)x = 0$$

if we make

$$y^{-1} = (n-1)^{-\frac{1}{n}} x^{-\frac{1}{n}} t, \qquad a = \frac{(n-1)^{\frac{n-1}{n}} x^{\frac{n-1}{n}}}{n}.$$

Hence, making $x = \epsilon^{\theta'}$ and representing $\frac{d}{d\theta'}$ by D', we have for the transformation of (IV)

$$\epsilon^{\theta} = \frac{(n-1)^{\frac{n-1}{n}}}{n} \epsilon^{\frac{n-1}{n}\theta'},$$

$$D = \frac{n}{n-1} D',$$

$$u = (n-1)^{-\frac{1}{n}} \epsilon^{\frac{-\theta'}{n}} t.$$

Substituting and multiplying the result by $\epsilon^{\frac{\theta'}{n}}$, we find

$$t - \left(\frac{n-1}{n}\right)^{n-1} \frac{\left[\frac{n}{n-1} D' - \frac{2n-1}{n-1}\right]^{n-1}}{[D']^{n-1}} \epsilon^{(n-1)\theta} t = 0,$$

which is Mr Harley's first equation.

If in (I) and (III) we make $1 - \frac{m}{n} = \alpha$, whence $m = n - n\alpha$, and at the same time change a into $ab^{-\frac{1}{n}}$, and y into $yb^{-\frac{1}{n}}$, we shall obtain the following somewhat more general statement of their united import.

The differential equations

$$u - \frac{1}{b} \frac{\left[\frac{n-1}{n} D - a\right]^{n-1} \left(\frac{D}{n} + a - 2\right)}{[D]^n} \epsilon^{n\theta} u = 0,$$

$$u - b \frac{[D-1]^n}{\left[\frac{n-1}{n} D + a - 1\right]^{n-1} \left(\frac{D}{n} - a + 1\right)} \epsilon^{n\theta} u = 0,$$

are both satisfied by the general integral

$$u = C_1 y_1^{n-na} + C_2 y_2^{n-na} \ldots + C_n y_n^{n-na},$$

when $y_1, y_2, \ldots y_n$ *are the roots of the algebraic equation*

$$y^n - a y^{n-1} + b = 0,$$

provided that for the first equation $a = \epsilon^{\theta}$, *and for the second* $a = \epsilon^{-\theta}$.

If $n = 2$, the above equations assume the forms

$$u - \frac{1}{b} \frac{\left(\frac{D}{2} - a\right)\left(\frac{D}{2} + a - 2\right)}{D(D-1)} \epsilon^{2\theta} u = 0,$$

$$u - b \frac{(D-1)(D-2)}{\frac{D^2}{4} - (a-1)^2} \epsilon^{2\theta} u = 0.$$

6. [The two principal papers by Mr Harley on the differential equations exhibited on page 190 are the following:

(1) On the Theory of the Transcendental Solution of Algebraic Equations, *Quarterly Journal of Mathematics,* Vol. v. pages 337...360.

(2) On a certain class of Linear Differential Equations. *Manchester Memoirs. Third Series.* Vol. II. pages 232...245.

In a letter bearing date January 13, 1864, Professor Boole pointed out to Mr Harley that his second equation might also be deduced from the general theorem discussed in Art. 5. Employing the above notation the deduction may be presented in the following form.

The equation (11) will reduce to

$$t^n - nt^{n-1} + (n-1)x = 0,$$

if we make

$$y = (n-1)^{-\frac{1}{n}} x^{-\frac{1}{n}} t, \qquad a = \frac{1}{n}(n-1)^{\frac{1}{n}} x^{\frac{1}{n}};$$

and for the transformation of (III) we have

$$\epsilon^{\theta} = \frac{1}{n}(n-1)^{\frac{1}{n}} \epsilon^{\frac{\theta'}{n}}, \qquad D = nD',$$

$$u = (n-1)^{-\frac{m}{n}} \epsilon^{-\frac{m}{n}\theta'} u'.$$

These substitutions being effected we arrive, after some slight reductions, at the following equation,

$$n^n [(n-1)D' - m]^{n-1} D'u' - (n-1)[nD' - m - 1]^n \epsilon^{\theta'} u' = 0,$$

which, making $m = 1$ and $u' = t$, gives

$$n^n [(n-1)D' - 1]^{n-1} D't - (n-1)[nD' - 2]^n \epsilon^{\theta'} t = 0,$$

an equation which admits of reduction. In fact, operating on both members with $(D' - 1)^{-1}$, and determining the constant, as in the former case, by the aid of the Lagrangean expansion, we find

$$n^{n-1}[(n-1)D']^{n-1} t - (nD' - n - 1)[nD' - 2]^{n-2} \epsilon^{\theta'} t = [n-1]^{n-1} \epsilon^{\theta'},$$

which is Mr Harley's second equation.

The references and deduction here given were to have been added to the memoir which is cited in page 189, according to Professor Boole's desire; but by some accident they were not printed, and the omission was not discovered until after his death.

Mr Harley has lately succeeded in obtaining the following extension of Professor Boole's theorem.

The differential equation

$$a^r \left[x \frac{d}{dx}\right]^n u$$

$$- \left[\frac{n-r}{n} x \frac{d}{dx} + \frac{m}{n} - 1\right]^{n-r} \left[\frac{r}{n} x \frac{d}{dx} - \frac{m}{n} - 1\right]^r x^r u = 0,$$

is satisfied by the m^{th} power of any root of the equation

$$y^n - xy^{n-r} + a = 0,$$

u being considered as a function of x.

From this he deduces the following; the differential equation

$$n^n \left[\frac{n-r}{r} x \frac{d}{dx} - \frac{m}{r}\right]^{n-r} \left[x \frac{d}{dx}\right]^r u$$

$$- (n-1)^r \left[\frac{n}{r} x \frac{d}{dx} - \frac{m}{r} - 1\right]^n x^r u = 0,$$

is satisfied by the m^{th} power of any root of the equation

$$y^n - ny^{n-r} + (n-1)\, x = 0.$$

For the materials of this Article I am indebted to Mr Harley.]

CHAPTER XXXI.

THE JACOBIAN THEORY OF THE LAST MULTIPLIER.

1. A SYSTEM of n differential equations of the first order and degree containing $n+1$ variables admits of n integrals of the form

$$u_1 = c_1, \quad u_2 = c_2, \ldots u_n = c_n,$$

$u_1, u_2, \ldots u_n$ being independent functions of the original variables. When $n-1$ of these integrals have been found they enable us to eliminate $n-1$ variables, with their differentials, from the given system of equations, and so to obtain a single final differential equation of the first order between the two remaining variables. The final equation admits of being made integrable by a factor, and its solution so found would constitute the n^{th} and last integral of the system. We propose in this Chapter to develope the theory of the above integrating factor as established by Jacobi. The term 'principle of the last multiplier,' which is more usually employed, seems objectionable; for the essence of Jacobi's discovery consisted not in demonstrating the existence or the nature of the last integrating factor, but in the peculiar form of the method which he gave for its determination, and in the relations which are implied in that form. The discovery may be briefly said to consist in this; *viz.* that instead of forming by means of the $n-1$ known integrals the final differential equation between two variables and applying methods analogous to those of Chap. V., to determine its integrating factor, we construct antecedently to all integration a linear partial differential equation of the first order, any one integral of which will enable us to assign an integrating factor of the final differential equation, *whatever the order of the previous integrations may have been.* Again, this partial differential equation depending for its construction only upon the form of the system given, we can often by examining it affirm beforehand that if all the integrals but one of the system be in any way found, the final integral will be deducible by quadratures. This happens in the case of the most important of all systems of differential equations—that of Dynamics.

Further, an ordinary differential equation of the n^{th} order being reducible to a system of n differential equations of the first order, Jacobi's theory may here also enable us to predicate the possibility of the last integration when the previous integrations have been effected.

Beginning with a single differential equation of the first order reduced to the form

$$\frac{dx}{X} = \frac{dy}{Y},$$

in which X and Y are functions of the two variables x and y, we know by Chap. v. that the integrating factor μ will be given by the solution of the partial differential equation

$$\frac{d(\mu X)}{dx} + \frac{d(\mu Y)}{dy} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the form of which should be carefully noticed.

Consider next a system of two differential equations of the first order expressed in the general form

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

X, Y, and Z being functions of the three variables x, y, z, and suppose one integral, represented by

$$\phi(x, y, z) = c \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

to be known. The function $\phi(x, y, z)$, or, as we shall express it for brevity, ϕ, will obviously satisfy the partial differential equation

$$X\frac{d\phi}{dx}+Y\frac{d\phi}{dy}+Z\frac{d\phi}{dz}=0 \quad\text{..............(4)},$$

of which indeed the given equations form the Lagrangean auxiliary system; see Chap. XIV.

If from the given integral we determine z as a function of x, y and c, and substitute its value in the first of the given differential equations, *viz.*

$$\frac{dx}{X}=\frac{dy}{Y},$$

the latter will be converted into a differential equation between x and y. But we may leave to the equation its prior form, provided that we regard X and Y as functions of the variables x and y, both explicitly as they appear therein, and implicitly as they are involved in z. And this being so, the equation (1) will become

$$\frac{d(\mu X)}{dx}+\frac{d(\mu X)}{dz}\frac{dz}{dx}+\frac{d(\mu Y)}{dy}+\frac{d(\mu Y)}{dz}\frac{dz}{dy}=0.$$

The values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ in this equation must be found from the known integral (3); they are

$$\frac{dz}{dx}=-\frac{d\phi}{dx}\div\frac{d\phi}{dz},\quad \frac{dz}{dy}=-\frac{d\phi}{dy}\div\frac{d\phi}{dz},$$

substituting which we have

$$\frac{d(\mu X)}{dx}\frac{d\phi}{dz}-\frac{d(\mu X)}{dz}\frac{d\phi}{dx}+\frac{d(\mu Y)}{dy}\frac{d\phi}{dz}-\frac{d(\mu Y)}{dz}\frac{d\phi}{dy}=0\text{...(5)}.$$

This then is the partial differential equation for determining μ. But the construction of this equation supposes ϕ to be known. We propose to shew that μ can be determined by a process in which the only partial differential equation to be solved can be constructed without the knowledge of ϕ.

Since by actual differentiation

$$\frac{d}{dx}\left(A\frac{d\phi}{dz}\right)-\frac{d}{dz}\left(A\frac{d\phi}{dx}\right)=\frac{dA}{dx}\frac{d\phi}{dz}-\frac{dA}{dz}\frac{d\phi}{dx},$$

it follows, writing μX for A, that

$$\frac{d(\mu X)}{dx}\frac{d\phi}{dz}-\frac{d(\mu X)}{dz}\frac{d\phi}{dx}=\frac{d}{dx}\left(\mu X\frac{d\phi}{dz}\right)-\frac{d}{dz}\left(\mu X\frac{d\phi}{dx}\right).$$

Similarly

$$\frac{d(\mu Y)}{dy}\frac{d\phi}{dz}-\frac{d(\mu Y)}{dz}\frac{d\phi}{dy}=\frac{d}{dy}\left(\mu Y\frac{d\phi}{dz}\right)-\frac{d}{dz}\left(\mu Y\frac{d\phi}{dy}\right).$$

Lastly, we have

$$0=\frac{d}{dz}\left(\mu Z\frac{d\phi}{dz}\right)-\frac{d}{dz}\left(\mu Z\frac{d\phi}{dz}\right).$$

Now adding the last three equations together we see that the first member of the result vanishes by (5): we have thus

$$\frac{d}{dx}\left(\mu X\frac{d\phi}{dz}\right)+\frac{d}{dy}\left(\mu Y\frac{d\phi}{dz}\right)+\frac{d}{dz}\left(\mu Z\frac{d\phi}{dz}\right)$$
$$-\frac{d}{dz}\left(\mu X\frac{d\phi}{dx}\right)-\frac{d}{dz}\left(\mu Y\frac{d\phi}{dy}\right)-\frac{d}{dz}\left(\mu Z\frac{d\phi}{dz}\right)=0.$$

The second line of the first member is equal to

$$-\frac{d}{dz}\left\{\mu\left(X\frac{d\phi}{dx}+Y\frac{d\phi}{dy}+Z\frac{d\phi}{dz}\right)\right\},$$

and therefore vanishes by (4). There remains then

$$\frac{d}{dx}\left(\mu X\frac{d\phi}{dz}\right)+\frac{d}{dy}\left(\mu Y\frac{d\phi}{dz}\right)+\frac{d}{dz}\left(\mu Z\frac{d\phi}{dz}\right)=0.$$

Hence if we put

$$\mu \frac{d\phi}{dz} = M,$$

we have

$$\frac{d(MX)}{dx} + \frac{d(MY)}{dy} + \frac{d(MZ)}{dz} = 0 \text{............(6).}$$

If then by the solution of this equation a value of M distinct from 0 be found, the function $\dfrac{M}{\dfrac{d\phi}{dz}}$ *will be an integrating factor of that final differential equation which remains when z has been eliminated from the system* (2) *by means of any known integral* $\phi = c$.

It will be observed that the equation for M is analogous in form to the equation for μ in the previous system. And this suggests the form of the general theorem.

Thus proceeding to the case of a system of three equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \frac{dt}{T},$$

we see that if

$$\psi(x, y, z, t) = c$$

be a known integral, ψ therefore satisfying the equation

$$X\frac{d\psi}{dx} + Y\frac{d\psi}{dy} + Z\frac{d\psi}{dz} + T\frac{d\psi}{dt} = 0, \text{........(7),}$$

then the system

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

will virtually involve only the variables x, y, z, since t

through the known integral becomes a function of x, y, z. The equation (6) now becomes

$$\frac{d(MX)}{dx} + \frac{d(MX)}{dt}\frac{dt}{dx} + \frac{d(MY)}{dy} + \frac{d(MY)}{dt}\frac{dt}{dy}$$

$$+ \frac{d(MZ)}{dz} + \frac{d(MZ)}{dt}\frac{dt}{dz} = 0,$$

or putting

$$\frac{dt}{dx} = -\frac{d\psi}{dx} \div \frac{d\psi}{dt}, \ldots$$

$$\frac{d(MX)}{dx}\frac{d\psi}{dt} - \frac{d(MX)}{dt}\frac{d\psi}{dx} + \frac{d(MY)}{dy}\frac{d\psi}{dt} - \frac{d(MY)}{dt}\frac{d\psi}{dy}$$

$$+ \frac{d(MY)}{dz}\frac{d\psi}{dt} - \frac{d(MY)}{dt}\frac{d\psi}{dz} = 0,$$

and this is equivalent to

$$\frac{d\left(MX\frac{d\psi}{dt}\right)}{dx} + \frac{d\left(MY\frac{d\psi}{dt}\right)}{dy} + \frac{d\left(MZ\frac{d\psi}{dt}\right)}{dz} + \frac{d\left(MT\frac{d\psi}{dt}\right)}{dt}$$

$$-\frac{d}{dt}\left\{M\left(X\frac{d\psi}{dx} + Y\frac{d\psi}{dy} + Z\frac{d\psi}{dz} + T\frac{d\psi}{dt}\right)\right\} = 0,$$

and therefore becomes on rejecting the term in the second line by (7), and putting

$$M\frac{d\psi}{dt} = N,$$

$$\frac{d(NX)}{dx} + \frac{d(NY)}{dy} + \frac{d(NZ)}{dz} + \frac{d(NT)}{dt} = 0 \ldots (8).$$

If from this equation a value of N distinct from 0 be obtained, then $M = \frac{N}{\frac{d\psi}{dt}}$, and therefore

$$\mu = \frac{N}{\frac{d\psi}{dt}\frac{d\phi}{dz}}.$$

This is the final multiplier, i.e. the integrating factor of the final differential equation between x and y which remains when z and t have been eliminated from the given system by means of the two known integrals. In calculating μ from the above formula we must proceed as follows. The value of $\frac{d\psi}{dt}$ must be found from any given integral $\psi = c$; but that of $\frac{d\phi}{dz}$ must be found from another integral from which by means of the former one t has been eliminated. Thus the general forms of the integrals will be

$$\psi(x, y, z, t) = c,$$

$$\phi(x, y, z, c) = c'.$$

Lastly, the values of $\frac{d\psi}{dt}$, $\frac{d\phi}{dz}$ found as above, and that of N given by any solution (distinct from 0) of the partial differential equation (8) having been substituted in the expression for μ, we must eliminate z and t from that expression by means of the two known integrals. The resulting function of x, y, c and c' will be the integrating factor sought.

The reasoning above employed is in its nature quite independent of the number of the equations of the original system. The general theorem to which it leads may be thus stated.

THEOREM. The system of n differential equations

$$\frac{dx}{X} = \frac{dy_1}{Y_1} = \frac{dy_2}{Y_2} = \dots \frac{dy_n}{Y_n}$$

being given, if a system of $n-1$ integrals

$$\phi_1 = c_1, \quad \phi_2 = c_2, \ldots. \phi_{n-1} = c_{n-1},$$

be so reduced by elimination that the variable y_1 shall not appear in ϕ_2, the variables y_1, y_2 shall not appear in ϕ_3, and so on, then the integrating factor μ of that final differential equation between x and y_n will be given by the formula

$$\mu = \frac{M}{\frac{d\phi_1}{dy_1}\frac{d\phi_2}{dy_2}\cdots\frac{d\phi_{n-1}}{dy_{n-1}}},$$

in which M represents any integral distinct from 0 of the partial differential equation

$$\frac{d(MX)}{dx} + \frac{d(MY_1)}{dy_1} \ldots. + \frac{d(MY_n)}{dy_n} = 0.$$

In applying this theorem the expression for μ must be freed from all the variables except x and y_n by means of the given integrals.

This is Jacobi's theorem. On account of its great importance I propose to give another demonstration of it founded upon the Calculus of Variations.

2. *Second demonstration founded upon the Calculus of Variations.*

It will be most convenient to present the proposed system of differential equations under the symmetrical form

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots. = \frac{dx_n}{X_n},$$

the independent variables being x_1, x_2, ... x_n of which X_1, X_2, ... X_n are any functions. We have thus $n-1$ differential equations, and we are to seek the integrating factor of the differential equation which remains when by means of $n-2$ known integrals $n-2$ of the variables with their differentials have been eliminated.

Suppose $P=c$ to be any integral of the system, then P satisfies, and it suffices that it satisfies, the partial differential equation

$$X_1\frac{dP}{dx_1}+X_2\frac{dP}{dx_2}\ldots\ldots+X_n\frac{dP}{dx_n}=0\ \ldots\ldots\ldots\ldots(a).$$

Now if in place of $x_1, x_2, \ldots\ldots x_n$ we introduce a new system of independent variables $u_1, u_2, \ldots\ldots u_n$ which are functions of the former, then we shall have

$$X_1\frac{dP}{dx_1}+X_2\frac{dP}{dx_2}\ldots\ldots+X_n\frac{dP}{dx_n}$$
$$=U_1\frac{dP}{du_1}+U_2\frac{dP}{du_2}\ldots\ldots+U_n\frac{dP}{du_n},$$

$U_1, U_2, \ldots\ldots U_n$ being functions of $u_1, u_2, \ldots\ldots u_n$. And by the theory of the transformation of multiple integrals,

$$\int^n\left(X_1\frac{dP}{dx_1}+X_2\frac{dP}{dx_2}\ldots\ldots+X_n\frac{dP}{dx_n}\right)dx_1\,dx_2\ldots\ldots dx_n$$
$$=\int^n\frac{1}{H}\left(U_1\frac{dP}{du_1}+U_2\frac{dP}{du_2}\ldots\ldots+U_n\frac{dP}{du_n}\right)du_1\,du_2\ldots\ldots du_n,$$

where
$$H=\begin{vmatrix}\dfrac{du_1}{dx_1},\ldots\ldots\ldots\dfrac{du_1}{dx_n}\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \dfrac{du_n}{dx_1},\ldots\ldots\ldots\dfrac{du_n}{dx_n}\end{vmatrix}.$$

The foregoing equation we may express in the form

$$\Sigma\int^n X_i\frac{dP}{dx_i}\,dx_1\,dx_2\ldots dx_n=\Sigma\int^n\frac{U_i}{H}\frac{dP}{du_i}\,du_1\,du_2\ldots\ldots du_n.$$

Hence, representing by δ an operation of differentiation which affects only the form of P as a function of $x_1, x_2, \ldots x_n$

or of $u_1, u_2, \ldots. u_n$, and not the independent variables themselves, we have

$$\Sigma\int^n X_i \frac{d\delta P}{dx_i} dx_1\, dx_2 \ldots. dx_n = \Sigma\int^n \frac{U_i}{H}\frac{d\delta P}{du_i} du_1\, du_2 \ldots du_n,$$

and therefore integrating by parts and equating the portions on each side which remain under the sign of n-fold integration,

$$\Sigma\int^n \frac{dX_i}{dx_i}\delta P\, dx_1\, dx_2 \ldots. dx_n$$
$$= \Sigma\int^n \frac{d}{du_i}\left(\frac{U_i}{H}\right)\delta P\, du_1\, du_2 \ldots. du_n.$$

Whence again transforming the integral in the first member

$$\Sigma\int^n \frac{dX_i}{dx_i}\delta P \frac{du_1\, du_2 \ldots. du_n}{H}$$
$$= \Sigma\int^n \frac{d}{du_i}\left(\frac{U_i}{H}\right)\delta P\, du_1\, du_2 \ldots. du_n,$$

and this being true quite irrespectively of the form of P, we have

$$\frac{1}{H}\Sigma\frac{dX_i}{dx_i} = \Sigma\frac{d}{du_i}\left(\frac{U_i}{H}\right).$$

In this equation Jacobi's theorem is virtually contained. For let the given equation be multiplied by any factor. Then changing in the above X_i into MX_i, and U_i into MU_i, we have

$$\frac{1}{H}\Sigma\frac{d(MX_i)}{dx_i} = \Sigma\frac{d}{du_i}\left(\frac{MU_i}{H}\right).$$

Hence, if M be determined to satisfy the equation

$$\Sigma\frac{d(MX_i)}{dx_i} = 0,$$

we shall have

$$\Sigma \frac{d}{du_i}\left(\frac{MU_i}{H}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

This is wholly independent of the relations connecting $u_1, u_2, \ldots u_n$ with $x_1, x_2, \ldots . x_n$. Now choose the $n-2$ variables $u_1, u_2, \ldots . u_{n-2}$ so that $u_1 = c_1$, $u_2 = c_2$, ... $u_{n-2} = c_{n-2}$ shall be integrals of the given partial differential equation (a). Then that equation transformed becomes

$$U_{n-1}\frac{dP}{du_{u-1}} + U_n \frac{dP}{du_n} = 0,$$

of which the auxiliary ordinary equation is

$$U_n du_{n-1} - U_{n-1} du_n = 0.$$

At the same time the equation (b) becomes

$$\frac{d}{du_{n-1}}\left(\frac{M}{H} U_{n-1}\right) + \frac{d}{du_n}\left(\frac{M}{H} U_n\right) = 0.$$

Hence $\frac{M}{H}$ is the integrating factor of the preceding differential equation between u_{n-1} and u_n.

Jacobi's theorem in its most general form is thus seen to be the following

THEOREM. If the system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} \ldots . = \frac{dx_n}{X_n}$$

be transformed by the introduction of a new system of variables $u_1, u_2, \ldots . u_n$, so chosen that

$$u_1 = c_1,\ u_2 = c_2, \ldots\ldots u_{n-2} = c_{n-2}$$

shall be integrals of the given system, then the final differential equation between u_{n-1} and u_n shall have for its integrating factor $\frac{M}{H}$, in which M is any function satisfying the partial differential equation

$$\frac{d(MX_1)}{dx_1} + \frac{d(MX_2)}{dx_2} \dots + \frac{d(MX_n)}{dx_n} = 0,$$

and H stands for the determinant

$$\left| \begin{array}{c} \frac{du_1}{dx_1}, \dots \dots \frac{du_1}{dx_n} \\ \dots\dots\dots\dots\dots \\ \frac{du_n}{dx_1}, \dots \dots \frac{du_n}{dx_n} \end{array} \right|.$$

The form of Jacobi's theorem obtained by the previous demonstration may be deduced from the above by choosing for u_{n-1}, u_n two of the original variables, for example x_{n-1}, x_n, and transforming the integrals u_1, u_2, ... u_{n-2} so that u_2 shall contain only $x_2 \dots x_n$, u_3 shall contain only $x_3 \dots x_n$, and so on.

Examples.

3. Jacobi has established by means of the above theorem the very remarkable theorem that in any ordinary dynamical problem the forces depending not upon the time but upon the material constitution of the system, if all the integrals but two of the dynamical equations are found, the two remaining integrals can be found by quadratures.

1st. In a dynamical system of free points the forces acting upon which depend only upon the position of the points, we have if we represent the entire system of rectangular coordinates taken in any order by $x, y, z, \dots$ and the corresponding resolved forces divided each by the corresponding mass by $X, Y, Z, \dots$ the system of equations

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \dots$$

or putting

$$\frac{dx}{dt} = x', \quad \frac{dy}{dt} = y', \ldots$$

$$dt = \frac{dx}{x'} = \frac{dy}{y'} \ldots = \frac{dx'}{X} = \frac{dy'}{Y} \ldots$$

Now as X, Y... do not contain t we may consider first the system

$$\frac{dx}{x'} = \frac{dy}{y'} \ldots = \frac{dx'}{X} = \frac{dy'}{Y} \ldots,$$

and it is evident that if we can find *all* the integrals of this system, t will be given by the equation

$$t = \int \frac{dx}{x'} + c,$$

x' having been first converted by means of the supposed integrals into a function of x.

To determine the last multiplier of the system last written we have first the equation

$$\frac{d(Mx')}{dx} + \frac{d(My')}{dy} \ldots + \frac{d(MX)}{dx'} + \frac{d(MY)}{dy'} \ldots = 0,$$

which since X, Y... do not contain x', y' ... is satisfied by $M =$ a constant. Giving to the constant the particular value 1, we see that if

$$u_1 = c_1, \; u_2 = c_2, \ldots \; u_{n-2} = c_{n-2}$$

are $n-2$ integrals of the system, and if by means of these we eliminate $n-2$ of the variables and construct the differential equation between the two remaining variables, the integrating factor of that equation will be $\frac{1}{H}$, in which H is the functional determinant of $u_1, u_2, \ldots u_n$.

2ndly. Suppose the system subject to a material connexion which establishes an equation of condition among some or all of the co-ordinates. If we represent the co-ordinates taken in any order and multiplied each by the square root of the corresponding mass by $x, y, \ldots$ the corresponding resolved forces by $X, Y, \ldots$ and the equation of condition expressed by means of the above modified co-ordinates by $\phi = 0$, the differential equations will be

$$\frac{d^2x}{dt^2} = X + \lambda \frac{d\phi}{dx}, \quad \frac{d^2y}{dt^2} = Y + \lambda \frac{d\phi}{dy}, \ldots,$$

the transformation above employed reducing all the equations to the same type. [See the next Chapter.]

Making

$$\frac{dx}{dt} = x', \quad \frac{dy}{dt} = y', \ldots$$

the system becomes

$$dt = \frac{dx}{x'} = \frac{dy}{y'} \ldots = \frac{dx'}{X + \lambda \frac{d\phi}{dx}} = \frac{dy'}{Y + \lambda \frac{d\phi}{dy}} \ldots,$$

and the Jacobian equation for M becomes

$$\frac{d(Mx')}{dx} + \frac{d(My')}{dy} \ldots + \frac{d\left\{M\left(X + \lambda \frac{d\phi}{dx}\right)\right\}}{dx'} + \frac{d\left\{M\left(Y + \lambda \frac{d\phi}{dy}\right)\right\}}{dy'} \ldots = 0.$$

Now ϕ does not contain $x', y' \ldots$. Let us inquire whether it is possible to determine M also as a function of $x, y \ldots$ without $x', y' \ldots$ so as to satisfy the above differential equation.

The equation would become

$$x'\frac{dM}{dx}+y'\frac{dM}{dy}+\ldots+M\left(\frac{d\lambda}{dx'}\frac{d\phi}{dx}+\frac{d\lambda}{dy'}\frac{d\phi}{dy}+\ldots.\right)=0,$$

or if we write

$$x'\frac{d}{dx}+y'\frac{d}{dy}\ldots.=\delta,$$

$$\delta M+M\left(\frac{d\lambda}{dx'}\frac{d\phi}{dx}+\frac{d\lambda}{dy'}\frac{d\phi}{dy}+\ldots.\right)=0,$$

and from this we must eliminate λ.

Now since $\phi=0$, we have by differentiating and putting $\frac{dx}{dt}=x', \ldots$

$$x'\frac{d\phi}{dx}+y'\frac{d\phi}{dy}+\ldots.=0,$$

and again differentiating

$$x'^2\frac{d^2\phi}{dx^2}+y'^2\frac{d^2\phi}{dy^2}+\ldots.+2x'y'\frac{d^2\phi}{dx\,dy}+\ldots$$

$$+\frac{d\phi}{dx}\frac{dx'}{dt}+\frac{d\phi}{dy}\frac{dy'}{dt}+\ldots.=0,$$

or since

$$\frac{dx'}{dt}=X+\lambda\frac{d\phi}{dx},\ldots$$

$$x'^2\frac{d^2\phi}{dx^2}+y'^2\frac{d^2\phi}{dy^2}+\ldots.+2x'y'\frac{d^2\phi}{dx\,dy}+\ldots.$$

$$+X\frac{d\phi}{dx}+Y\frac{d\phi}{dy}+\ldots.$$

$$+\lambda\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\ldots.\right\}=0;$$

and differentiating with respect to x',

$$2\left(x'\frac{d^2\phi}{dx^2}+y'\frac{d^2\phi}{dx\,dy}+\ldots.\right)+\frac{d\lambda}{dx'}\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\ldots.\right\}=0,$$

or if we make

$$\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\ldots.=Q,$$

$$2\delta\frac{d\phi}{dx}+\frac{d\lambda}{dx'}Q=0.$$

Similarly

$$2\delta\frac{d\phi}{dy}+\frac{d\lambda}{dy'}Q=0,$$

....

Therefore

$$2\frac{d\phi}{dx}\,\delta\,\frac{d\phi}{dx}+2\frac{d\phi}{dy}\,\delta\,\frac{d\phi}{dy}+\ldots.$$

$$+\left(\frac{d\lambda}{dx'}\frac{d\phi}{dx}+\frac{d\lambda}{dy'}\frac{d\phi}{dy}+\ldots.\right)Q=0,$$

or

$$\delta Q+\left(\frac{d\lambda}{dx'}\frac{d\phi}{dx}+\frac{d\lambda}{dy'}\frac{d\phi}{dy}+\ldots.\right)Q=0,$$

and now eliminating $\frac{d\lambda}{dx'}\frac{d\phi}{dx}+\frac{d\lambda}{dy'}\frac{d\phi}{dy}+\ldots$

we obtain $$Q\delta M-M\delta Q=0,$$

which is satisfied by $M=Q$.

4. [Among Professor Boole's manuscripts I found five pages in German, forming part of a memoir, which was probably intended for Crelle's *Mathematical Journal*. The memoir was to have discussed two applications of the Calcu-

lus of Variations; one to the Jacobian Theory of the Last Multiplier, and the other to the Solution of Pfaff's equation

$$X_1\,dx_1 + X_2\,dx_2 + \ldots\ldots + X_{2n}\,dx_{2n} = 0.$$

But there is only a single paragraph relating to the second application.

The manuscript contains the same demonstration of the Jacobian Theory of the Last Multiplier as in Art. 2 of the present Chapter; after this demonstration some remarks occur of which the substance will now be given.]

It is worthy of notice, that Jacobi in the 36th volume of Crelle's *Journal*, deduced by the aid of the Calculus of Variations the result on which the preceding demonstration of the Theory of the Last Multiplier depends. In fact, he shewed that if V denotes any function of

$$x_1,\ x_2,\ \ldots\ x_n,\ z,\ \frac{dz}{dx_1},\ \ldots\ \frac{dz}{dx_n},$$

and V be transformed by the introduction of a new system of independent variables $u_1, u_2, \ldots u_n$, then the following relation holds,

$$\Delta\left(\frac{dV}{dz} - \frac{d}{dx_1}\frac{dV}{d\frac{dz}{dx_1}} - \ldots - \frac{d}{dx_n}\frac{dV}{d\frac{dz}{dx_n}}\right)$$

$$= \frac{d(\Delta V)}{dz} - \frac{d}{du_1}\frac{d(\Delta V)}{d\frac{dz}{du_1}} - \ldots - \frac{d}{du_n}\frac{d(\Delta V)}{d\frac{dz}{du_n}},$$

where
$$\Delta = \begin{vmatrix} \dfrac{dx_1}{du_1}, & \ldots\ldots \dfrac{dx_1}{du_n} \\ \ldots\ldots & \ldots\ldots \\ \dfrac{dx_n}{du_1}, & \ldots\ldots\ \dfrac{dx_n}{du_n} \end{vmatrix}$$

Jacobi applies this result to the transformation of the expression

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2}.$$

But neither Jacobi himself, nor any other person, so far as I know, has drawn attention to the application of the result which I have given here.

[The substance of the single paragraph relating to the second application of the Calculus of Variations will now be given.]

Clebsch has earned the thanks of all who are interested in the higher parts of the Theory of Differential Equations, since he has performed the same service for Pfaff's problem as Jacobi did for the Theory of Partial Differential Equations of the first order, and thereby for the equations of Dynamics. But while I recognise the great importance of the results, I consider it desirable to give a simpler deduction of the system of partial differential equations therein involved, and on which the other results depend.

CHAPTER XXXII.

THE DIFFERENTIAL EQUATIONS OF DYNAMICS.

[It will be seen that this is only a fragment of the Chapter which was to have appeared under this title.]

I do not propose in this Chapter to discuss the origin and interpretation of the differential equations of motion or to enter into those details of their application which are found in all ordinary treatises on Dynamics. But they constitute a system analytically so remarkable from the forms in which it is capable of being expressed, and from the general methods of integration which emerge out of those forms, that they are well deserving of a special attention.

Referred to rectangular co-ordinates the differential equations for the motion of a system of points free or connected are

$$m\frac{d^2x}{dt^2}=X+\lambda\frac{d\phi}{dx}+\mu\frac{d\psi}{dx}\cdots$$

$$m\frac{d^2y}{dt^2}=Y+\lambda\frac{d\phi}{dy}+\mu\frac{d\psi}{dy}\cdots$$

$$m\frac{d^2z}{dt^2}=Z+\lambda\frac{d\phi}{dz}+\mu\frac{d\psi}{dz}\cdots$$

$$m'\frac{d^2x'}{dt^2}=X'+\lambda\frac{d\phi}{dx'}+\mu\frac{d\psi}{dx'}\cdots.$$

....

Here m is the mass at the point (x, y, z), m' that at (x', y', z'), X, Y, Z the resolved forces at (x, y, z) tending severally to increase those co-ordinates, and so on. Lastly

$$\phi = 0, \quad \psi = 0, \dots$$

are the equations of condition each of which may involve all the co-ordinates, and λ, μ... are indeterminate multipliers.

The above is usually termed the first Lagrangean form of the differential equations. In applying it we must either eliminate λ, μ... from the given equations, and then by the equations of condition just so many of the co-ordinates with their differentials, or we must retain λ, μ,... as variables so conditioned that the values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$,... in the system shall satisfy identically the differential equations involving $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$,.... derived from $\phi = 0$, $\psi = 0$,... viz. the equations

$$\frac{d^2\phi}{dt^2} = 0, \quad \frac{d^2\psi}{dt^2} = 0, \dots$$

The first Lagrangean system may by a slight transformation be reduced to a form in which all the equations are of one type, viz. of the type which they would have if all the masses were equal to unity.

For taking the first equation of the system and dividing by $m^{\frac{1}{2}}$ we may express the result in the form

$$\frac{d^2(m^{\frac{1}{2}}x)}{dt^2} = \frac{X}{m^{\frac{1}{2}}} + \lambda \frac{d\phi}{d(m^{\frac{1}{2}}x)} + \mu \frac{d\psi}{d(m^{\frac{1}{2}}x)} \dots$$

from which we see that if x, y ... had been taken to represent the entire system of co-ordinates taken in any order and multiplied each by the square root of the corresponding mass, and X, Y... the corresponding resolved forces taken in the same order and *divided* each by the square root of the corresponding mass, the system of equations would have been

$$\frac{d^2x}{dt^2} = X + \lambda\frac{d\phi}{dx} + \mu\frac{d\psi}{dx}\ldots$$

$$\frac{d^2y}{dt^2} = Y + \lambda\frac{d\phi}{dy} + \mu\frac{d\psi}{dy}\ldots$$

....

all being of one type. In general investigations this form is to be preferred.

From the first Lagrangean form another known as the second Lagrangean, and from this again a third known as the Hamiltonian are derived. The second Lagrangean form is properly speaking an expression for the effect of a transformation of co-ordinates in the most general sense upon the original system, i.e. of a transformation which in place of $x, y, \ldots$ the entire system of given co-ordinates substitutes a new system of variables $\xi, \eta, \ldots$ the expressions of which as functions of $x, y, \ldots$ are known. It is not necessary that this new system of variables should be co-ordinates in the proper sense of that term, determining three by three the positions of the several masses; it suffices that they should in their entirety determine and be determined by the co-ordinates given.

The second Lagrangean form may be established as follows:

Differentiating the equations $\phi = 0$, $\psi = 0, \ldots$ with respect to any one of the new variables ξ we have

$$\frac{d\phi}{dx}\frac{dx}{d\xi} + \frac{d\phi}{dy}\frac{dy}{d\xi}\ldots = 0,$$

$$\frac{d\psi}{dx}\frac{dx}{d\xi} + \frac{d\psi}{dy}\frac{dy}{d\xi}\ldots = 0;$$

whence if we multiply the equations of the given system by $\frac{dx}{d\xi}, \frac{dy}{d\xi}, \ldots$ and add, we have

$$\frac{dx}{d\xi}\frac{d^2x}{dt^2} + \frac{dy}{d\xi}\frac{d^2y}{dt^2}\ldots = X\frac{dx}{d\xi} + Y\frac{dy}{d\xi}\ldots$$

CHAPTER XXXIII.

ON THE PROJECTION OF A SURFACE ON A PLANE.

[THE following memoir was found among Professor Boole's manuscripts; a Title and Introductory Remarks were to have been prefixed, but with this exception the memoir appears to be finished for publication. It is sufficiently connected with the subject of Differential Equations to find a place in the present volume.

The memoir by Sir John Herschel to which allusion is made is entitled, *On a new Projection of the Sphere;* this was read before the Royal Geographical Society of London on the 11th of April, 1859, and was printed as part of the Journal of the Society, Vol. XXX. 1860, pages 100...106. A chart of the World on Sir John Herschel's projection has been published by A. and C. Black of Edinburgh.

The history of the subject will be found in Chapter XXIII. of the *Coup d'œil historique sur la Projection des Cartes de Géographie*... Par M. D'Avezac, Paris, 1863.

For the materials of this introductory notice I am indebted to Sir John Herschel.]

1. Let x, y, z be the rectangular co-ordinates of any point on the given surface; x', y' the co-ordinates of the corresponding point on the plane of projection. Let the equation of the given surface be

$$F(x, y, z) = 0;$$

or, for simplicity,

$$F = 0.$$

The condition of projection upon which Sir John Herschel's investigations are founded, and which we shall adopt here, is that of the similarity of corresponding infinitesimal areas on the surface and on the plane. The object of the problem then in general is the discovery of the mode in which x', y' depend upon x, y, and z in accordance with the above condition; its object in any particular case is the determination of x', y' as functions of x, y, z.

Regarding then x', y' as ultimately functions of x, y, z we have

$$dx' = \frac{dx'}{dx}\,dx + \frac{dx'}{dy}\,dy + \frac{dx'}{dz}\,dz,$$

$$dy' = \frac{dy'}{dx}\,dx + \frac{dy'}{dy}\,dy + \frac{dy'}{dz}\,dz,$$

in which dx, dy, dz are not independent, but are connected by the condition

$$\frac{dF}{dx}\,dx + \frac{dF}{dy}\,dy + \frac{dF}{dz}\,dz = 0.$$

Now for brevity write

$$\frac{dx'}{dx} = a, \quad \frac{dx'}{dy} = b, \quad \frac{dx'}{dz} = c,$$

$$\frac{dy'}{dx} = a', \quad \frac{dy'}{dy} = b', \quad \frac{dy'}{dz} = c',$$

$$\frac{dF}{dx} = A, \quad \frac{dF}{dy} = B, \quad \frac{dF}{dz} = C;$$

then

$$dx' = a\,dx + b\,dy + c\,dz \;\ldots\ldots\ldots\ldots\ldots\ldots \;(1),$$

$$dy' = a'dx + b'dy + c'dz \;\ldots\ldots\ldots\ldots\ldots\ldots \;(2),$$

$$0 = A\,dx + B\,dy + C\,dz \;\ldots\ldots\ldots\ldots\ldots\ldots \;(3).$$

Now the condition of the similarity of infinitesimal corresponding areas may be resolved into the two following conditions, viz.:

1st. The equality of their corresponding angles.

2ndly. The proportionality of their corresponding sides.

And these conditions we shall introduce separately.

1st. Assuming any point x', y' on the plane of projection, let x' alone vary, and the infinitesimal line generated is dx', while (since $dy'=0$) (2) and (3) become

$$a'dx+b'dy+c'dz=0,$$

$$Adx+Bdy+Cdz=0,$$

whence, if we write

$$L=Bc'-Cb', \quad M=Ca'-Ac', \quad N=Ab'-Ba',$$

we have
$$\frac{dx}{L}=\frac{dy}{M}=\frac{dz}{N} \qquad \text{(4)},$$

so that the direction cosines of the infinitesimal line on the surface F corresponding to the line dx' on the plane (x', y') will be

$$\frac{L}{(L^2+M^2+N^2)^{\frac{1}{2}}}, \quad \frac{M}{(L^2+M^2+N^2)^{\frac{1}{2}}}, \quad \frac{N}{(L^2+M^2+N^2)^{\frac{1}{2}}} \qquad \text{(5)}.$$

In like manner, if y' alone vary, we shall find for the direction cosines of the infinitesimal line on the surface F which corresponds to dy' on the plane

$$\frac{L'}{(L'^2+M'^2+N'^2)^{\frac{1}{2}}}, \quad \frac{M'}{(L'^2+M'^2+N'^2)^{\frac{1}{2}}}, \quad \frac{N'}{(L'^2+M'^2+N'^2)^{\frac{1}{2}}} \qquad \text{(6)},$$

where $L'=Bc-Cb$, $M'=Ca-Ac$, $N'=Ab-Ba$.

By the first of the conditions of similarity the angle between these lines on the surface must be a right angle since

dx' and dy' are at right angles. Hence we have, from (5) and (6),

$$LL' + MM' + NN' = 0 \quad \text{.............................(7).}$$

2ndly. The ratio of the length of the element dx' to the corresponding element on the surface is

$$\frac{dx'}{\sqrt{dx^2 + dy^2 + dz^2}},$$

or, by (1),

$$\frac{adx + bdy + cdz}{\sqrt{dx^2 + dy^2 + dz^2}},$$

and therefore by (4)

$$\frac{aL + bM + cN}{\sqrt{L^2 + M^2 + N^2}},$$

equating which to the corresponding expression for the ratio of the length of dy' to that of its projection on the surface, we have

$$\frac{aL + bM + cN}{\sqrt{L^2 + M^2 + N^2}} = \frac{a'L' + b'M' + c'N'}{\sqrt{L'^2 + M'^2 + N'^2}} \quad \text{.......... (8).}$$

Now if we substitute for L, M, N, L', M', N' their values, we shall find

$$aL + bM + cN = A(b'c - bc') + B(c'a - ca') + C(a'b - ab'),$$

$$a'L' + b'M' + c'N' = A(bc' - b'c) + B(ca' - c'a) + C(ab' - a'b),$$

and the second members of these equations differ only in sign.

Thus (8) may be expressed in the form

$$\left\{A(b'c - bc') + B(c'a - ca') + C(ab' - a'b)\right\}$$
$$\times \left\{\frac{1}{(L^2 + M^2 + N^2)^{\frac{1}{2}}} - \frac{1}{(L'^2 + M'^2 + N'^2)^{\frac{1}{2}}}\right\} = 0 \ldots (9).$$

But the first factor of the first member of this equation being the determinant of the system

$$adx + bdy + cdz = 0,$$

$$a'dx + b'dy + c'dz = 0,$$

$$Adx + Bdy + Cdz = 0,$$

expresses when equated to zero the condition that if in the system (1), (2), (3) dy' vanishes dx' shall also vanish; and dx' and dy' being independent, this condition cannot be satisfied, so that (9) reduces to

$$\frac{1}{(L^2 + M^2 + N^2)^{\frac{1}{2}}} - \frac{1}{(L'^2 + M'^2 + N'^2)^{\frac{1}{2}}} = 0,$$

whence

$$L'^2 + M'^2 + N'^2 - L^2 - M^2 - N^2 = 0 \quad \text{.........(10)},$$

and this, with (7), will fully express the conditions of similarity.

2. If we multiply (7) by $2\sqrt{-1}$, and add and subtract the result from (10), we obtain the equivalent system

$$\left.\begin{aligned}(L' + L\sqrt{-1})^2 + (M' + M\sqrt{-1})^2 + (N' + N\sqrt{-1})^2 = 0\\ (L' - L\sqrt{-1})^2 + (M' - M\sqrt{-1})^2 + (N' - N\sqrt{-1})^2 = 0\end{aligned}\right\} \text{..(11).}$$

Now
$$L' \pm L\sqrt{-1} = \frac{dF}{dy}\frac{dx'}{dz} - \frac{dF}{dz}\frac{dx'}{dy}$$

$$\pm \left(\frac{dF}{dy}\frac{dy'}{dz} - \frac{dF}{dz}\frac{dy'}{dy}\right)\sqrt{-1}$$

$$= \frac{dF}{dy}\frac{d(x' \pm y'\sqrt{-1})}{dz} - \frac{dF}{dz}\frac{d(x' \pm y'\sqrt{-1})}{dy}.$$

Writing then

$$x' + y'\sqrt{-1} = u, \quad x' - y'\sqrt{-1} = v,$$

we have

$$L' + L\sqrt{-1} = \frac{dF}{dy}\frac{du}{dz} - \frac{dF}{dz}\frac{du}{dy},$$

$$L' - L\sqrt{-1} = \frac{dF}{dy}\frac{dv}{dz} - \frac{dF}{dz}\frac{dv}{dy}.$$

In the same way

$$M' + M\sqrt{-1} = \frac{dF}{dz}\frac{du}{dx} - \frac{dF}{dx}\frac{du}{dz},$$

$$M' - M\sqrt{-1} = \frac{dF}{dz}\frac{dv}{dx} - \frac{dF}{dx}\frac{dv}{dz},$$

$$N' + N\sqrt{-1} = \frac{dF}{dx}\frac{du}{dy} - \frac{dF}{dy}\frac{du}{dx},$$

$$N' - N\sqrt{-1} = \frac{dF}{dx}\frac{dv}{dy} - \frac{dF}{dy}\frac{dv}{dx}.$$

Substituting which in the system (11) there result

$$\left.\begin{aligned}\left(\frac{dF}{dy}\frac{du}{dz} - \frac{dF}{dz}\frac{du}{dy}\right)^2 + \left(\frac{dF}{dz}\frac{du}{dx} - \frac{dF}{dx}\frac{du}{dz}\right)^2 \\ + \left(\frac{dF}{dx}\frac{du}{dy} - \frac{dF}{dy}\frac{du}{dx}\right)^2 = 0 \\ \left(\frac{dF}{dy}\frac{dv}{dz} - \frac{dF}{dz}\frac{dv}{dy}\right)^2 + \left(\frac{dF}{dz}\frac{dv}{dx} - \frac{dF}{dx}\frac{dv}{dz}\right)^2 \\ + \left(\frac{dF}{dx}\frac{dv}{dy} - \frac{dF}{dy}\frac{dv}{dx}\right)^2 = 0\end{aligned}\right\} \ldots\ldots(12),$$

to which we may give the somewhat more convenient form

$$\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right\}$$
$$-\left(\frac{dF}{dx}\frac{du}{dx}+\frac{dF}{dy}\frac{du}{dy}+\frac{dF}{dz}\frac{du}{dz}\right)^2=0\ldots\ldots(\text{I}).$$

$$\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}\left\{\left(\frac{dv}{dx}\right)^2+\left(\frac{dv}{dy}\right)^2+\left(\frac{dv}{dz}\right)^2\right\}$$
$$-\left(\frac{dF}{dx}\frac{dv}{dx}+\frac{dF}{dy}\frac{dv}{dy}+\frac{dF}{dz}\frac{dv}{dz}\right)^2=0\ldots(\text{II}).$$

These are partial differential equations of the first order, serving to determine u and v as functions of x, y, z.

But it is not necessary to solve the equations in their general form. For, x, y, and z being connected by the equation of the surface, the above equations may always be so reduced as to involve only two independent variables. As latitude and longitude determine the position of a point on the earth, so two co-ordinates of any given species will determine the position of a point on the given surface, and these co-ordinates, when fixed upon, become the independent variables of the problem.

Let s and t represent such co-ordinates, and let their expressions in terms of x, y, z give

$$s=\phi_1(x, y, z),\quad t=\phi_2(x, y, z),$$

which equations combined with that of the given surface will reciprocally determine x, y, z as functions of s and t. Then 1st the differential coefficients of F which in the equations (I), (II), are functions of x, y, z may be transformed into functions of s and t; 2ndly, we have

$$\frac{du}{dx}=\frac{du}{ds}\frac{ds}{dx}+\frac{du}{dt}\frac{dt}{dx},$$
$$\frac{du}{dy}=\frac{du}{ds}\frac{ds}{dy}+\frac{du}{dt}\frac{dt}{dy},$$
$$\frac{du}{dz}=\frac{du}{ds}\frac{ds}{dz}+\frac{du}{dt}\frac{dt}{dz},$$

and as $\frac{ds}{dx}$, $\frac{dt}{dx}$... are known functions of x, y, z, they also are expressible in terms of s and t. The result of these substitutions will then be to convert (I) into a partial differential equation in which u is the dependent and s and t the independent variables, and this equation being, like (I), of the first order and second degree in the differential coefficients of u, will be of the form

$$P\left(\frac{du}{ds}\right)^2 + Q\frac{du}{ds}\frac{du}{dt} + R\left(\frac{du}{dt}\right)^2 = 0.$$

For v we shall have an exactly similar equation with the same coefficients.

The above equation is, by the solution of a quadratic, resolvable into two equations of the form

$$\frac{du}{ds} - \lambda_1\frac{du}{dt} = 0, \quad \frac{du}{ds} - \lambda_2\frac{du}{dt} = 0.$$

To these correspond the respective auxiliary equations

$$dt + \lambda_1 ds = 0, \quad dt + \lambda_2 ds = 0 \text{ .... } (13).$$

If the integrals of these are

$$S = c_1, \quad T = c_2,$$

respectively, then we have

$$u = \phi(S), \quad u = \psi(T).$$

Now v being determinable by an equation of the same form as u, it follows that of the above two values of u one must be assigned to v, so that the solution of the problem will be contained in the system

$$u = \phi(S), \quad v = \psi(T),$$

or in the system

$$u = \phi(T), \quad v = \psi(S).$$

The particular forms of the arbitrary functions ϕ and ψ will depend solely upon the nature of the problem under consideration.

One other point remains to be noticed. The first members of (12) are essentially positive, being composed of squares; so are then the first members of (I), (II); and so, if the intermediate transformations are real, is the first member of the equation whose coefficients are P, Q, R. Hence the quadratic determining λ_1, λ_2 will have imaginary roots of the form $\alpha \pm \beta\sqrt{-1}$. Ultimately therefore it will suffice to integrate one equation of the system (13) and then to deduce the solution of the other by changing $\sqrt{-1}$ into $-\sqrt{-1}$.

3. *Application of the above formulæ when the given surface is an oblate spheroid, such as the earth.*

Let the plane of the equator be that of projection, the centre being the origin. Let the co-ordinates x, y pass through the meridians of 0 and of 90° respectively, and z through the poles. The equation of the surface will be

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \qquad\qquad (14),$$

where a is the earth's equatorial, b its polar radius. Let also the latitude of the point x, y, z be represented by s, the longitude by t. We have

$$F=\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}-1,$$

$$\frac{dF}{dx}=\frac{2x}{a^2},\qquad \frac{dF}{dy}=\frac{2y}{a^2},\qquad \frac{dF}{dz}=\frac{2z}{b^2},$$

and substituting in (I),

$$\left(\frac{x^2+y^2}{a^4}+\frac{z^2}{b^4}\right)\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right\}$$
$$-\left(\frac{x}{a^2}\frac{du}{dx}+\frac{y}{a^2}\frac{du}{dy}+\frac{z}{b^2}\frac{du}{dz}\right)^2=0,$$

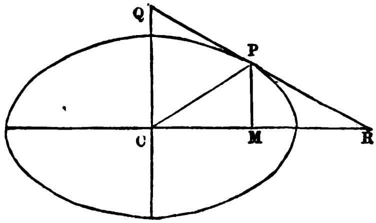
or, if we represent $\frac{a^2}{b^2}$ by h^2,

$$(x^2+y^2+h^4z^2)\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right\}$$
$$-\left(x\frac{du}{dx}+y\frac{du}{dy}+h^2z\frac{du}{dz}\right)^2=0\ldots\ldots\ldots(15).$$

Now as x, y are rectangular co-ordinates in the plane of the equator, and x passes through the first meridian, we have

$$\frac{y}{x}=\tan t.$$

Again, representing in the annexed figure the meridian of the point P, or (x, y, z) touched by the straight line QR in the same plane, we have $CM=\sqrt{x^2+y^2}$, $MP=z$. Therefore if $\sqrt{x^2+y^2}=r$, the equation of the meridian is



$$\frac{r^2}{a^2}+\frac{z^2}{b^2}=1,$$

that of the tangent

$$\frac{rr'}{a^2}+\frac{zz'}{b^2}=1,$$

r', z' being current rectangular co-ordinates of the tangent. Hence

$$\tan CQR=\frac{a^2z}{b^2r}=\frac{h^2z}{\sqrt{x^2+y^2}}.$$

But $CQR=$ latitude. Therefore finally

$$s=\tan^{-1}\frac{h^2z}{\sqrt{x^2+y^2}},\quad t=\tan^{-1}\frac{y}{x}\ldots\ldots\ldots(16),$$

and we must now transform (15) so as to make s and t the independent variables.

From the above equations combined with (14) we find

$$x=\frac{ah\cos t}{\sqrt{h^2+\tan^2 s}},\quad y=\frac{ah\sin t}{\sqrt{h^2+\tan^2 s}},\quad z=\frac{a\tan s}{h\sqrt{h^2+\tan^2 s}}\ \dots\ (17),$$

and substituting in (15),

$$\sec^2 s\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right\}$$

$$-\left(\cos t\frac{du}{dx}+\sin t\frac{du}{dy}+\tan s\frac{du}{dz}\right)^2=0\ \dots\dots\ (18).$$

Again,

$$\frac{du}{dx}=\frac{du}{ds}\frac{ds}{dx}+\frac{du}{dt}\frac{dt}{dx},$$

$$\frac{du}{dy}=\frac{du}{ds}\frac{ds}{dy}+\frac{du}{dt}\frac{dt}{dy},$$

$$\frac{du}{dz}=\frac{du}{ds}\frac{ds}{dz}+\frac{du}{dt}\frac{dt}{dz}.$$

Now $\dfrac{ds}{dx}=\dfrac{-h^2zx}{\sqrt{x^2+y^2}\,(x^2+y^2+h^4z^2)}=\dfrac{-\sin s\cos s\cos t\sqrt{H}}{ah}$,

where $H=h^2+\tan^2 s$. In like manner

$$\frac{ds}{dy}=\frac{-\sin s\cos s\sin t\sqrt{H}}{ah},$$

$$\frac{ds}{dz}=\frac{h\cos^2 s\sqrt{H}}{a},$$

$$\frac{dt}{dx}=\frac{-\sin t\sqrt{H}}{ah},$$

$$\frac{dt}{dy}=\frac{\cos t\sqrt{H}}{ah},$$

$$\frac{dt}{dz}=0.$$

Hence

$$\frac{du}{dx}=\frac{\sqrt{H}}{ah}\left(-\sin s\cos s\cos t\frac{du}{ds}-\sin t\frac{du}{dt}\right),$$

$$\frac{du}{dy}=\frac{\sqrt{H}}{ah}\left(-\sin s\cos s\sin t\frac{du}{ds}+\cos t\frac{du}{dt}\right),$$

$$\frac{du}{dz}=\frac{\sqrt{H}}{ah}\left(h^2\cos^2 s\frac{du}{ds}\right).$$

Substituting these values in (18), and dividing by the common factor $\frac{H}{a^2h^2}$, we have on reduction

$$\left(\frac{du}{dt}\right)^2+\cos^2 s\,\{1+(h^2-1)\cos^2 s\}^2\left(\frac{du}{ds}\right)^2=0,$$

which is resolvable into

$$\frac{du}{dt}-\sqrt{-1}\cos s\,\{1+(h^2-1)\cos^2 s\}\frac{du}{ds}=0,$$

$$\frac{du}{dt}+\sqrt{-1}\cos s\,\{1+(h^2-1)\cos^2 s\}\frac{du}{ds}=0,$$

partial differential equations of which the integrals are included in the common formula

$$u=\phi\left(\int\frac{ds}{\cos s\,\{1+(h^2-1)\cos^2 s\}}\pm t\sqrt{-1}\right).$$

Now $$\int\frac{ds}{\cos s\{1+(h^2-1)\cos^2 s\}}$$

$$=\int\frac{ds}{\cos s}+(1-h^2)\int\frac{\cos s\,ds}{1+(h^2-1)\cos^2 s}$$

$$=\int\frac{ds}{\cos s}+(1-h^2)\int\frac{\cos s\,ds}{h^2-(h^2-1)\sin^2 s}.$$

$$=\int \frac{ds}{\cos s} - e^2 \int \frac{\cos s\, ds}{1 - e^2 \sin^2 s}, \quad \left(\text{since } e^2 = \frac{a^2 - b^2}{a^2}\right)$$

$$= \log \tan\left(\frac{\pi}{4} + \frac{s}{2}\right) + \frac{e}{2} \log \frac{1 - e \sin s}{1 + e \sin s}$$

$$= \log \left\{\left(\frac{1 - e \sin s}{1 + e \sin s}\right)^{\frac{e}{2}} \tan\left(\frac{\pi}{4} + \frac{s}{2}\right)\right\}.$$

Hence

$$u = \phi\left[\log\left\{\left(\frac{1 - e \sin s}{1 + e \sin s}\right)^{\frac{e}{2}} \tan\left(\frac{\pi}{4} + \frac{s}{2}\right)\right\} \pm t\sqrt{-1}\right],$$

or, changing $\phi(t)$ into $\phi(\epsilon^t)$,

$$u = \phi\left\{\left(\frac{1 - e \sin s}{1 + e \sin s}\right)^{\frac{e}{2}} \tan\left(\frac{\pi}{4} + \frac{s}{2}\right) \epsilon^{\pm t\sqrt{-1}}\right\},$$

$$v = \psi\left\{\left(\frac{1 - e \sin s}{1 + e \sin s}\right)^{\frac{e}{2}} \tan\left(\frac{\pi}{4} + \frac{s}{2}\right) \epsilon^{\mp t\sqrt{-1}}\right\}.$$

Let r and θ be the polar co-ordinates of that point in the plane of projection which corresponds to the point whose latitude and longitude on the surface are s and t; and let

$$S = \left(\frac{1 - e \sin s}{1 + e \sin s}\right)^{\frac{e}{2}} \tan\left(\frac{\pi}{4} + \frac{s}{2}\right),$$

then the complete solution assumes the very simple form

$$r\epsilon^{\theta\sqrt{-1}} = \phi\left(S\epsilon^{\pm t\sqrt{-1}}\right), \quad r\epsilon^{-\theta\sqrt{-1}} = \psi\left(S\epsilon^{\mp t\sqrt{-1}}\right) \ldots\ldots\text{(III)}.$$

Of particular deductions the most interesting is that which arises from the supposition that the parallels of latitude are projected into circles round the pole. This requires that r

should be independent of t, a condition which is satisfied in the most general manner by assuming

$$\phi(t) = Ct^n, \quad \psi(t) = C't^n,$$

we then find

$$r\epsilon^{\theta\sqrt{-1}} = CS^n\epsilon^{\pm nt\sqrt{-1}}, \quad r\epsilon^{-\theta\sqrt{-1}} = C'S^n\epsilon^{\mp nt\sqrt{-1}},$$

whence, on multiplication and division,

$$r^2 = CC'S^{2n}, \quad \epsilon^{2\theta\sqrt{-1}} = \frac{C}{C'}\epsilon^{\pm 2nt\sqrt{-1}},$$

whence, A and B being new arbitrary constants derived from C and C'

$$r = AS^n, \quad \theta = \pm nt + B.$$

If we observe that θ and t should vanish together, we have $B = 0$, and the equation $\theta = \pm nt$ shews that the surface of the sphere will be projected into a sector of a circle, the arc of which is to the circumference of the circle as $n : 1$. Thus, if $n = \frac{1}{4}$, the sphere is projected upon a quadrant, and so on.

The other equation gives

$$r = A\left\{\tan\left(\frac{\pi}{4} + \frac{s}{2}\right)\right\}^n \left(\frac{1 - e\sin s}{1 + e\sin s}\right)^{\frac{ne}{2}}.$$

If $s = 0$ we find $r = A$, whence A is the distance of the equator from the pole in the plane of projection, and if that distance, which is arbitrary, be assumed as the unit, we have

$$r = \tan\left\{\left(\frac{\pi}{4} + \frac{s}{2}\right)\right\}^n \left(\frac{1 - e\sin s}{1 + e\sin s}\right)^{\frac{ne}{2}}$$

for the distance from the pole of that parallel whose latitude is s. We may give to this expression a better form by assuming $p = \frac{\pi}{2} + s$, and introducing an auxiliary quantity q determined by the equation

$$e \cos p = \cos q.$$

We have then

$$r = \left(\tan \frac{p}{2}\right)^n \left(\cot \frac{q}{2}\right)^{ne}.$$

The following table gives the values of r for the sphere and for the spheroid whose eccentricity is ·08 (which is about that of the earth), for each ten degrees of polar distance, for the values $n = 1$, and $n = \frac{1}{4}$.

Polar Distance.	$n=1$		$n=\frac{1}{4}$	
	Sphere.	Spheroid.	Sphere.	Spheroid.
10°	·0875	·0880	·5439	·5447
20°	·1763	·1774	·6480	·6490
30°	·2679	·2694	·7195	·7205
40°	·3640	·3658	·7767	·7777
50°	·4663	·4682	·8264	·8272
60°	·5774	·5792	·8717	·8724
70°	·7002	·7017	·9148	·9153
80°	·8391	·8400	·9571	·9574
90°	1·0000	1·0000	1·0000	1·0000
100°	1·1918	1·1904	1·0448	1·0445
110°	1·4281	1·4250	1·0932	1·0926
120°	1·7321	1·7265	1·1472	1·1463
130°	2·1445	2·1357	1·2101	1·2089
140°	2·7475	2·7340	1·2875	1·2859
150°	3·7321	3·7114	1·3899	1·3880
160°	5·6713	5·6372	1·5432	1·5409
170°	11·4301	11·3581	1·8387	1·8358

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